CPS102 DISCRETE MATHEMATICS

Practice Final Exam

In contrast to the homework, no collaborations are allowed. You can use all your notes, calculator, and any books you think are useful. Write legibly and formulate each answer concisely, using only the space provided on this handout.

Your name: ______________________________

	credit	max
Question 1		10
Question 2		10
Question 3		10
Question 4		10
Question 5		10
Question 6		10
Question 7		10
Question 8		10
Question 9		10
Question 10		10
Total		100

Question 1.

Prove or disprove: there exists a prime $p > 3$ such that $p + 2$ and $p + 4$ are also prime.

Hint: False. p is prime, in this case, p is odd. $p + 2$ and $p + 4$ are both odd. Then $p \pmod 3 = 1$ or 2. If $p \pmod 3 = 1$, then $p + 2 \pmod 3 = 0$ and $p + 2$ is not prime. If $p \pmod 3 = 2$, then $p + 4 \pmod 3 = 1$, then $p + 4$ is not prime.

Question 2.

Give tight asymptotic bounds (in terms of Θ) for the following recurrence using the master theorem.

$$(1) \qquad T(n) = 16T(n/4) + \log^2 n$$

Or try the easier version

$$(2) \qquad T(n) = 16T(n/4) + n^2$$

Hint:
(1) is harder than what is intended. Try some exercises from the textbook instead.
(1) requires another version of Master Theorem which states as follows:

$$T(n) = aT(n/b) + f(n)$$

if $f(n) = O(n^{log_b a - \epsilon})$ for some constant $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$. Therefore, $a = 16$, $b = 4$ and $f(n) = O(n^{2-\epsilon})$, therefore

$$T(n) = \Theta(n^2).$$

(2) based on Master Theorem directly, $T(n) = \Theta(n^2 \log n)$.

Question 3.

Function $A(x, y, z)$ for $x, y, z \geq 0$ is defined as follows:

1. $A(x, y, 0) = y + 1$
2. $A(x, 0, 1) = x$
3. $A(x, 0, 2) = 0$
4. $A(x, 0, z + 3) = 1$
5. $A(x, y, z) = A(x, A(x, y - 1, z), z - 1)$

Prove that

(a) $A(x, y, 1) = x + y$

(b) $A(x, y, 2) = xy$

(c) $A(x, y, 3) = x^y$

Hint: (a) Induction on y.
Base case: $y = 0$, $A(x, 0, 1) = x$ by (2).
Inductive case: assume $A(x, y, 1) = x + y$ then
$A(x, y + 1, 1) = A(x, A(x, y, 1), 0) = A(x, x + y, 0) = x + y + 1$ by (5), (1) and inductive hypothesis.
(b) induction on y.
Base case: $y = 0$, $A(x, 0, 2) = 0$ by (3)
Inductive case: assume $A(x, y, 2) = xy$ then
$A(x, y + 1, 2) = A(x, A(x, y, 2), 1) = A(x, xy, 1) = xy + x = x(y + 1)$ by (5), inductive hypothesis and previous result.
(c) Induction on y.
Base case: $y = 0$, $A(x, 0, 3) = 1 = x^0$ by (4).
Inductive case: Assume $A(x, y, 3) = x^y$, then
$A(x, y + 1, 3) = A(x, A(x, y, 3), 2) = A(x, x^y, 2) = xx^y = x^{y+1}$ by (5), (a) and inductive hypothesis.

Question 4.

We draw cards from an ordinary deck of 52 playing cards. The cards are to be drawn successively at random and without replacement. What is the probability that the second heart appears on the fifth draw?

Hint:
A : event that 1 heart appears in the first 4 cards drawn.
B : event that it is a heart in the 5th draw.
We need to compute $P(A \cap B)$.

$$P(A) = \binom{13}{1}\binom{39}{3} \Big/ \binom{52}{4}$$

$$P(B|A) = 12/48$$

Then compute $P(A \cap B) = P(A)P(B|A) = ...$

Question 5.

A tournament is a simple directed graph such that if u and v are distinct vertices in the graph, exactly one of (u, v) and (v, u) is an edge of the graph. Assume all vertices are labeled. How many different tournaments are there with n vertices?

Hint: There are $\binom{n}{2} = n(n-1)/2$ edges in a tournament. Orient each edge with 2 possible directions, the answer is $2^{n(n-1)/2}$.

Question 6.

We hash n keys into $k = 1000$ memory locations one by one. What is the probability that the first i records do not produce a collision? Assume each key is independently and uniformly hashed into the memory locations.

Hint: The probability that the i-th location does not has a collision is $\frac{1000-(i-1)}{1000}$. The probability to be computed is $\frac{1000}{1000}\frac{999}{1000}\ldots\frac{1000-(i-1)}{1000}$.

Question 7.

A pair of dice is rolled in a remote location and an honest observer informs us that at least one of the dice came up six. What is the probability that the sum of the numbers that came up on the two dice is seven, given the information provided by the honest observer?

Hint: use (i, j) to represent that the first dice comes up i and the second dice comes up j. Note that there are 36 equally likely outcomes.
S : event that at least one dice comes up 6.
T : even that the sum of dice is 7.
Compute

$$P(T|S) = P(S \cap T)/P(S) = 2/11$$

where $P(S \cap T) = 2/36$ and $P(S) = 1 - P(\bar{S}) = 1 - 25/36 = 11/36$.

Question 8.

Give a formula for the coefficient of x^k in the expansion of $(x - 1/x)^{100}$ where k is an integer.

Hint: By Binomial Theorem.

$$\binom{100}{j} x^{100-j}(-1/x)^j = \binom{100}{j} x^{100-2j}(-1)^j$$

for all integers k such that $k = 100 - 2j$, the coefficient is $\binom{100}{(100-k)/2}(-1)^{(100-k)/2}$.

Question 9.

Use Fermat's Little Theorem to compute 3^{302} mod 5.

Hint: Based on Fermat's Little Theorem,

$$3^4 = 1 \pmod 5$$

$$3^{300} = (3^4)^{75} = 1^{75} = 1 \pmod 5$$
$$3^{302} = 3^2 3^{300} = 9 \pmod 5 = 4.$$

Question 10.

True or false, are the following logically equivalent? $\Longleftrightarrow$ represents equivalence.

(a) $\neg(p \oplus q) \Longleftrightarrow (p \Leftrightarrow q)$.

(b) $\neg(p \Leftrightarrow q) \Longleftrightarrow (\neg p \Leftrightarrow q)$.

(c) $[(p \Rightarrow q) \Rightarrow r] \Longleftrightarrow [p \Rightarrow (q \Rightarrow r)]$.

(d) $[\neg p \Rightarrow (q \Rightarrow r)] \Longleftrightarrow [q \Rightarrow (p \vee r)]$.

Hint: From (a) to (d), True, True, False, True. Using truth tables.