COMPSCI 638: Graph Algorithms	October 25, 2019
Lecture 18	
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1 Overview

In the last lecture, we inrdocued the Laplacian matrix of an undirected graph. In this lecture, we continue studying the Laplacian by examining its relationship with electrical flows.

2 Electrical Flows

Recall that in spectral graph theory, we consider the Laplacian matrix of an undirected graph $G = (V, E)$ on n vertices and m edges. The Laplacian $L \in \mathbb{R}^{n\times n}$ of G is defined as follows:

$$L_{ij} = \begin{cases} \deg(i) & \text{if } i = j \\ -1 & \text{if } i \neq j \text{ and } (i,j) \in E \\ 0 & \text{otherwise.} \end{cases}$$

We now let s, t be vertices of G and consider a particular flow (see Lecture 1) from s to t. This *electrical flow* views each edge e as a resistor with resistance 1. (If e has weight $w(e)$, then the resistance is $1/w(e)$; for simplicity, we assume the edges are not weighted.)

To produce our flow, we attach a 1-volt battery that sends current from s to t through G. This induces a *current* $i \in \mathbb{R}^m$ and voltage *potentials* $v \in \mathbb{R}^n$. Also, let $r \in \mathbb{R}^m$ denote the *resistance* vector; recall that we assume $r(a, b) = 1$ for every $(a, b) \in E$. Note that we have fixed an arbitrary orientation of the edges, so if $i(a, b) < 0$, then current flows from b to a.

The current (i.e., flow) values produced by this procedure can be determined via Ohm's Law (see below), Kirchoff's Voltage Law (the potential changes in any cycle sum to zero), and Kirchoff's Current Law (current is conserved at every node). Applying these laws yields a set of linear equations whose solution gives the current value $i(a, b)$ along every edge (a, b).

Ohm's Law: Adopting the convention that current flows from high voltage to low voltage, Ohm's Law states that for any edge $(a, b) \in E$, we have

$$i(a,b) = \frac{1}{r(a,b)}(v(a) - v(b)) = v(a) - v(b).$$

To write this more compactly, let $B \in \mathbb{R}^{m\times n}$ denote the *signed edge-vertex adjacency matrix* of G, defined as follows:

$$B_{(i,j),u} = \begin{cases} 1 & \text{if } u = i \\ -1 & \text{if } u = j \\ 0 & \text{otherwise.} \end{cases}$$

Then notice that Ohm's Law can be written as $i = Bv$, and the Laplacian is $L = B^\top B$.

External currents: Now let $i_{ext} \in \mathbb{R}^n$ denote vector containing the value of current leaving every vertex. By conservation of current at vertex a, we have

$$i_{ext}(a) = \sum_{b:(a,b) \in E} i(a,b),$$

and again, we can write this in matrix form as $i_{ext} = B^\top i = B^\top Bv = Lv$.

Effective resistance: Let $a, b \in V$ and suppose we send one unit of current into a and remove one unit of current from b. The potential difference $v(a) - v(b)$ needed to realize this flow is the *effective resistance* between a and b, and we denote it by $R(a,b)$. To compute $R(a,b)$, notice that the current is $\chi_a - \chi_b$, where $\chi_u \in \{0,1\}^n$ is 1 in the u-th coordinate and 0 everywhere else.

From flow conservation (see above), we have $v = L^{-1}(\chi_a - \chi_b)$, and to extract the value of $v(a) - v(b)$, we can premultiply v by $(\chi_a - \chi_b)$:

$$R(a,b) = (\chi_a - \chi_b)^\top v = (\chi_a - \chi_b)^\top L^{-1}(\chi_a - \chi_b). \tag{1}$$

Note that L is not necessarily invertible, but for ease of presentation, we assume L^{-1} exists. One way to avoid this problem is to use the *Moore-Penrose pseudo-inverse* of L; we omit the details.

Energy: The *energy* of a flow $i \in \mathbb{R}^m$ is defined as follows:

$$\mathcal{E}(i) = \sum_{(a,b) \in E} i(a,b)^2 \cdot r(a,b) = \sum_{(a,b) \in E} \frac{(v(a) - v(b))^2}{r(a,b)}.$$

We can relate energy and effective resistance by the following fact. Intuitively, it states that if we replace the entire graph with a single edge (s,t) with resistance $R(s,t)$, then the total energy in the system remains the same as long as the value of the flow is one.

Fact 1. *The energy of a unit-value electrical s-t flow is equal to the effective resistance between s and t.*

Proof. Recall from (1) that

$$R(s,t) = (\chi_s - \chi_t)^\top L^{-1}(\chi_s - \chi_t).$$

Furthermore, since our external flow vector is $\chi_s - \chi_t$, it is equal to Lv. Substituting yields

$$R(s,t) = (Lv)^\top L^{-1}(Lv) = v^\top L^T L^{-1} Lv = v^\top Lv,$$

where the final equality holds because the Laplacian matrix is symmetric. Finally, as we saw in Lecture 17, we have

$$v^\top Lv = \sum_{(a,b) \in E} (v(a) - v(b))^2 = \mathcal{E}(i),$$

where the final equality holds due to our assumption that every edge has unit resistance. □

The next relationship between energy and effective resistance is known as Thompson's Principle. In essence, combined with Fact 1, it states that among all s-t flows with unit value, the one with the minimum amount of energy is precisely the electrical flow.

Theorem 2 (Thompson's Principle)**.** *For any (arbitrary) s-t flow j with unit value,* $\mathcal{E}(j) \geq R(s,t)$.

Proof. Let i be the unit-value electrical s-t flow and let B denote the signed edge-vertex adjacency matrix of G. Notice that $B^\top i = B^\top j = \chi_s - \chi_t$, so $B^\top(j - i) = 0$. In other words, the flow $c = j - i$ is a *circulation*, because the flow at *every* vertex (including s and t) satisfies flow conservation.

Now if we decompose j as $j = i + c$, we get

$$\mathcal{E}(j) = \sum_{(a,b)\in E} j(a,b)^2 = \sum_{(a,b)\in E} (i(a,b) + c(a,b))^2 = \mathcal{E}(i) + 2\sum_{(a,b)\in E} i(a,b)\cdot c(a,b) + \mathcal{E}(c).$$

Since energies are always non-negative, it suffices to prove that the summation term in the final expression above is also non-negative. By Ohm's Law, we have $i(a,b) = v(a) - v(b)$, so

$$\begin{aligned}\sum_{(a,b)\in E} i(a,b)\cdot c(a,b) &= \sum_{(a,b)\in E} (v(a) - v(b))\cdot c(a,b)\\ &= \sum_{(a,b)\in E} v(a)\cdot c(a,b) + v(b)\cdot c(a,b)\\ &= \sum_{a\in V} v(a) \sum_{b:(a,b)\in E} c(a,b),\end{aligned}$$

where the final equality uses the fact that $c(a,b) = -c(b,a)$. Since c is a circulation, each term in this final sum is equal to zero, so $\mathcal{E}(j) = \mathcal{E}(i) + \mathcal{E}(c) \geq \mathcal{E}(i)$. □

Finally, we use Thompson's principle to prove Rayleigh's Monotonicity Principle, which states an intuitive relationship between the resistance values in a network and the effective resistance between any two vertices. For any network G', we let r' and R' respectively denote the resistances and effective resistances in G'.

Theorem 3 (Rayleigh's Monotonicity Principle). *Let $G' = (V, E)$ be a network such that $r'(a,b) \geq r(a,b)$ for every edge $(a,b) \in E$. Then $R'(s,t) \geq R(s,t)$ for every $s,t \in V$.*

Proof. Let i, i' denote unit-value electrical s-t flows in G, G' respectively. Then from Fact 1, we have $R'(s,t) = \mathcal{E}_{G'}(i')$ and $R(s,t) = \mathcal{E}_G(i)$, where $\mathcal{E}_H(j)$ denotes the energy of flow j in network H. Thus, it suffices to prove the following:

$$\mathcal{E}_{G'}(i') \geq \mathcal{E}_G(i') \geq \mathcal{E}_G(i).$$

The first inequality holds because decreasing resistance values decreases energy, by the definition of energy. The second inequality holds because Thompson's Principle implies i is the energy-minimizing unit-value s-t flow in G. □

Conductance and cut value: For any $a, b \in V$, let $C(a,b) = 1/R(a,b)$ denote the *conductance* between a and b. We now look at the relationship between $C(a,b)$ and $\lambda(a,b)$, the connectivity of a and b. Recall that $\lambda(a,b)$ is defined as the minimum cut (or maximum flow) value from a to b.

Fact 4. *For any $a, b \in V$, $C(a,b) \leq \lambda(a,b)$.*

Proof. Consider a cut S separating $a \in S$ and $b \in V \setminus S$ containing $\lambda(a,b)$ edges. Now suppose we contract all vertices of S into a and all vertices of $V \setminus S$ into b, resulting in a graph G' with two vertices and $\lambda(a,b)$ parallel edges.

Notice that in G', the effective resistance $R'(a,b)$ is equal to $1/\lambda(a,b)$ due to current conservation. By Rayleigh's Monotonicity Principle, we also know $R(a,b) \geq R'(a,b)$ because contracting two endpoints of an edge reduces its resistance to zero. Thus, $1/C(a,b) \geq 1/\lambda(a,b)$, and rearranging this proves the desired claim. □

3 Summary

In this lecture, we introduced the concept of electrical flows and saw their relationship with the Laplacian of a graph. We also proved various facts effective resistances and the energy of a flow.