Lecture 13 Shortest Path, Minimum Spanning Tree

Thursday, February 28, 2019 2:47 PM

- shortest path with negative edge length
  - arbitrage example

    $(u_1, u_2, \cdots, u_k)$

    1 unit of u_i $\quad C(u_i,u_{i+1})$: 1 unit of u_{i+1} $= C(u_i,u_{i+1})$ unit of u_i

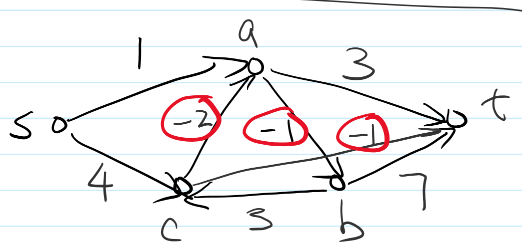
    exchanging 1 unit of u_1 to u_k using this path $\underline{\frac{1}{C(u_1,u_2)} \cdot \frac{1}{C(u_2,u_3)} \cdots \frac{1}{C(u_{k-1},u_k)}}$

    if we define $w(u,v) = \log C(u,v)$ then

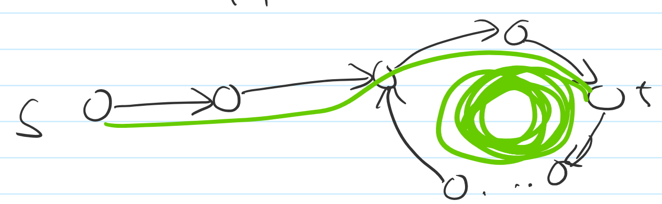
    $$\log \frac{1}{C(u_1,u_2)} \cdot \frac{1}{C(u_2,u_3)} \cdots \frac{1}{C(u_{k-1},u_k)} = \underbrace{-\sum_{i=1}^{k-1} w(u_i,u_{i+1})}_{\text{length of path}}$$

  - shortest path and negative cycles

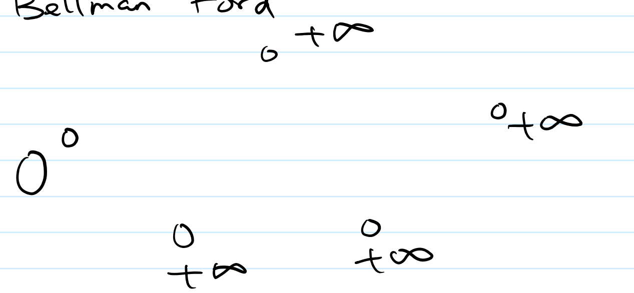
    Claim: if s can reach a negative cycle $(u_1, u_2, \cdots, u_k)$ then the shortest path from s to any u_i is not defined.



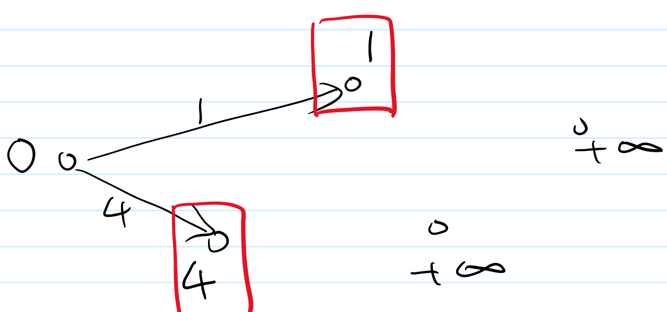
negative cycle $\sum_{i=1}^{k-1} w(u_i,u_{i+1}) + w(u_k,u_1) < 0$



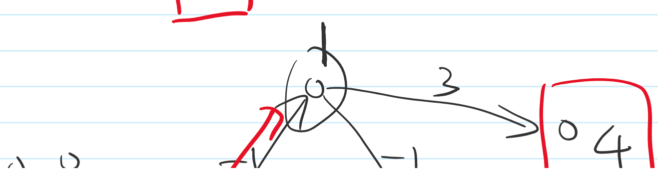
- Bellman Ford



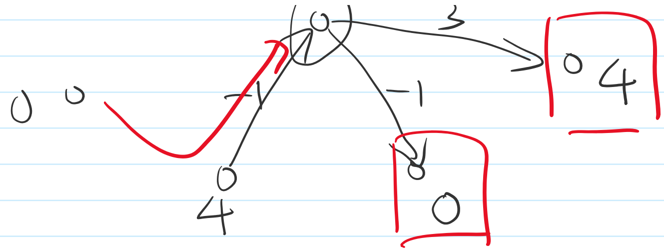
$d(u,0)$: length of shortest path from s to u using at most 0 edges



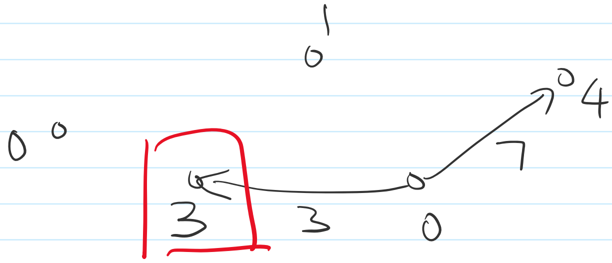
$d(u,1)$: ______ using at most 1 edge



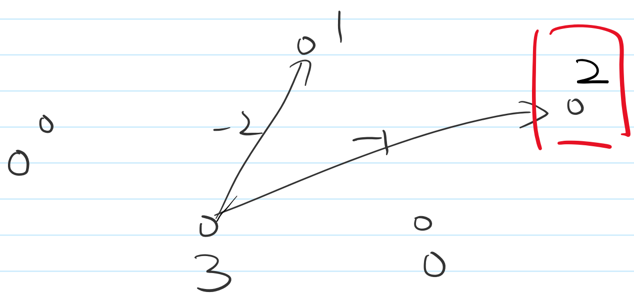
$d(u,2)$: ______ using at most 2 edges



$d(u,3)$

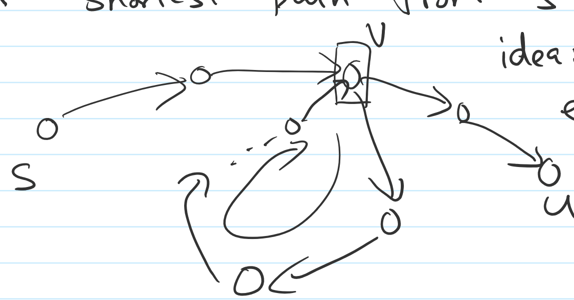


$d(u,4)$

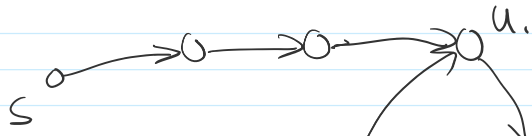


$$\boxed{d(u,5) = d(u,4)}$$

- Theorem: If the graph does not have any negative cycle, then $d(u, n-1)$ will be the shortest path distance from s to u. also, $d(u,n) = d(u,n-1)$ for any vertex u.
  on the other hand, if there is a negative cycle reachable from s, there exists vertex u s.t. $d(u,n) < d(u,n-1)$
- Proof: ① prove that $d(u,i)$ is indeed length of shortest path from s to u using at most i edges.
  ② Claim: shortest path from s to u has length $\leq n-1$
  idea: show path will visit each vertex at most once.



③



$$\boxed{w(u_1,u_2)+w(u_2,u_3)+\cdots+w(u_{K-1},u_K)+w(u_K,u_1)} < 0$$

$$d(u_{i+1},n) \le d(u_i,n-1)+w(u_i,u_{i+1})$$

$$d(u_1,n) \le d(u_K,n-1)+w(W_K,1)$$

take the sum

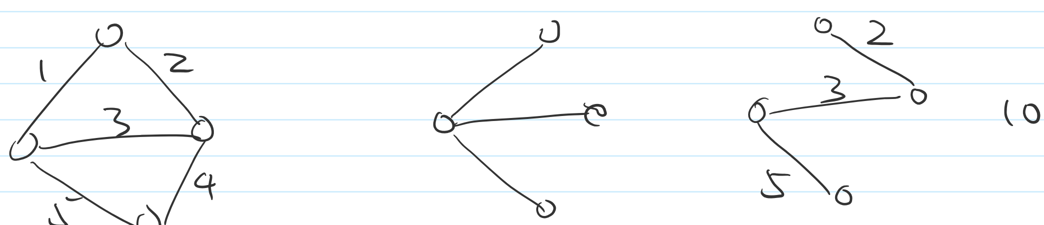
$$\sum_{i=1}^{n} d(u_i,n) \le \sum_{i=1}^{n} d(u_i,n-1) + \underbrace{\text{length of cycle}}_{<0}$$

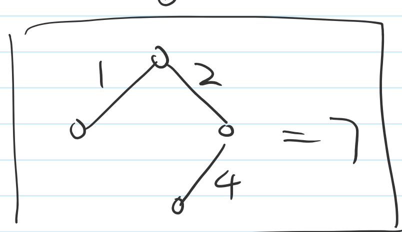
$$< \sum_{i=1}^{n} d(u_i,n-1) \qquad \square$$

- running time: $O(nm)$

- minimum spanning tree
  - spanning tree

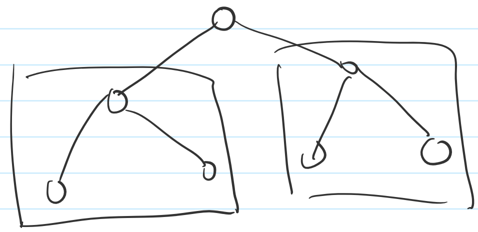
    a spanning tree of a graph $G=(V,E)$ is a subset of $n-1$ edges in E, such that all pairs of vertices are connected by these edges

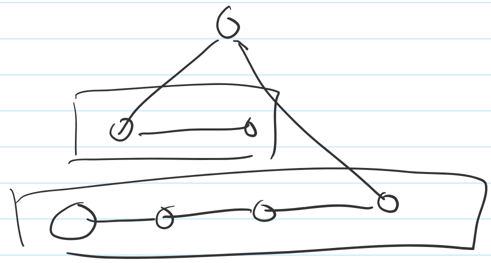




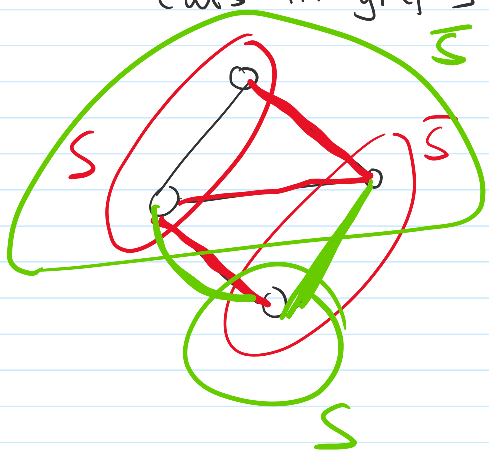
  - properties for MST

    subtrees of MST are also MST





- cuts in graphs



construct a cut: choose a set of vertices S

call the remaining vertices $\bar{S}$

$$C(S,\bar{S}) = \{(u,v)\in E \mid u\in S, v\in \bar{S}\}$$

if $(u,v)\in E$, $u\in S$, $v\in \bar{S}$ say edge (u,v) "crosses" the cut $(S,\bar{S})$

- for any spanning tree T, any cut $(S,\bar{S})$

  there must be at least one edge (u,v) in both T and $(S,\bar{S})$