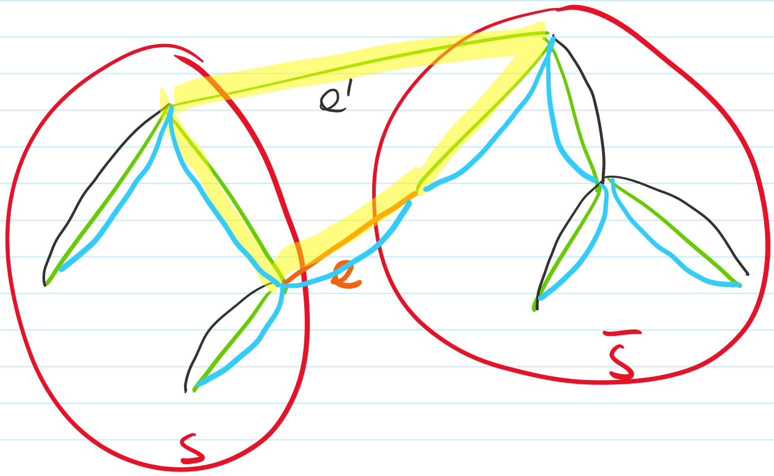
Lecture 14 Minimum Spanning Trees

Tuesday, March 5, 2019 2:35 PM

- Key Lemma:
  - Given a subset of edges F, suppose F is a subset of edges of a minimum spanning tree T. Pick any cut $(S, \bar{S})$ that does not intersect with F, let e be an edge with minimum cost in this cut $(S, \bar{S})$ then $F \cup \{e\}$ is a subset of edges in a minimum spanning tree T'.

    (T' may not be equal to T)



green: minimum spanning tree that we want to find. (T)

black: edges in F

: cycle C

blue: new minimum spanning tree T'

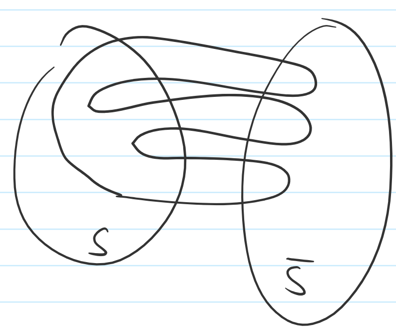
Proof: in case 1, e is actually an edge in T, this case is easy because $F \cup \{e\} \subseteq T$, can choose $T' = T$.

in case 2, e is not an edge in T

adding e to T will form a cycle, call it C

we know cycle C must intersect with cut $(S, \bar{S})$ in an even number of edges.

so there must be another edge $e' \in C$, e' also crosses cut $(S, \bar{S})$



we will swap e and e'

define $T' = (T \setminus \{e'\}) \cup \{e\}$

$$\text{cost}(T') = \text{cost}(T) - \underline{w(e')} + \underline{w(e)}$$

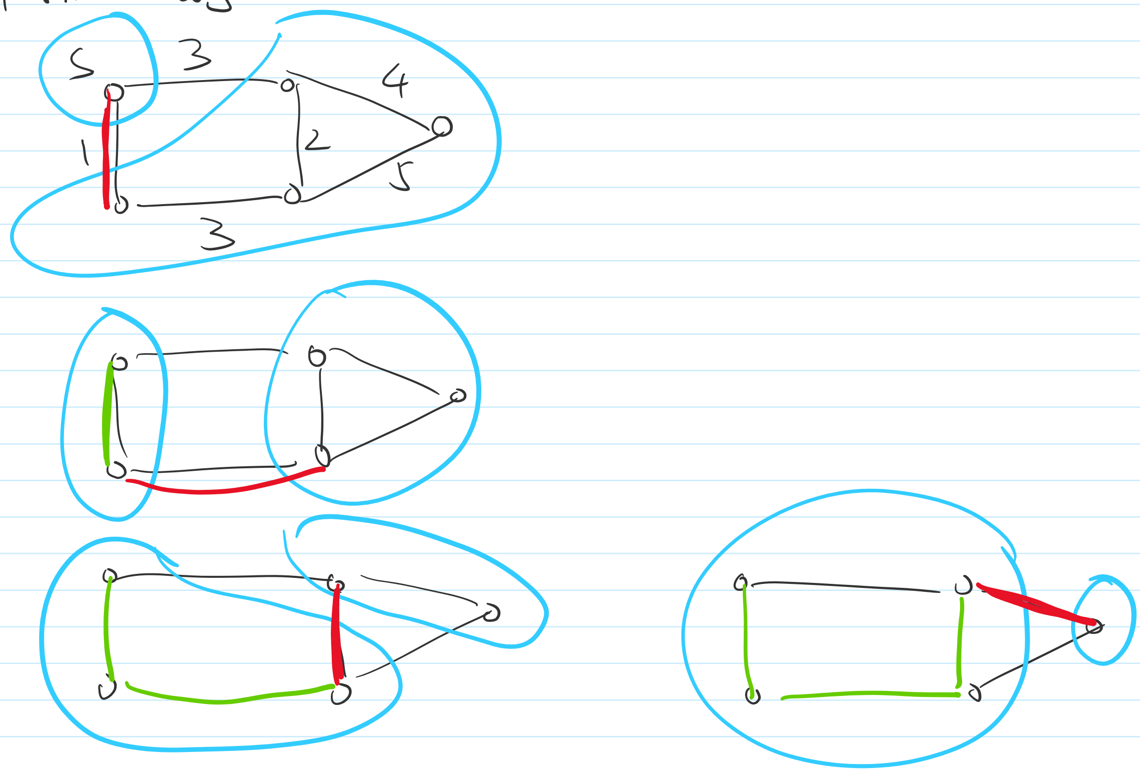
$$\leq \text{cost}(T)$$

by assumption $w(e) \leq w(e')$

this means T' is a MST, also, $F \cup \{e\} \subseteq T'$, this concludes the proof $\square$

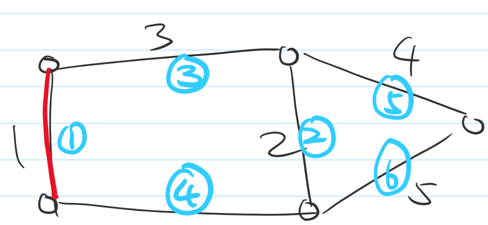
- Proof of general algorithm
  - Induction Hypothesis: At iteration i, the edges selected by the alg is a subset of some MST.
    - Base case: $i=0$, set of edges selected is empty
    - Induction step: Key lemma

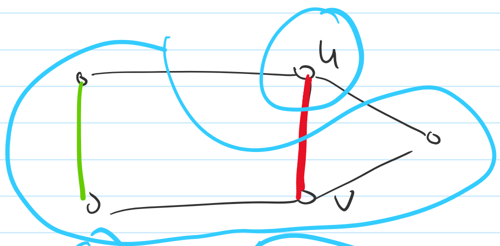
- Prim's algorithm

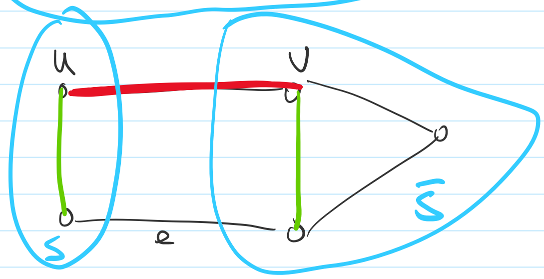


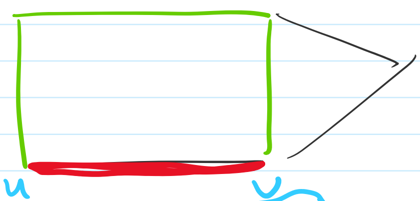
- in implementation maintain dis[u]
  dis[u]: minimum cost of an edge (u,v) where v is a visited vertex
  (Only maintained if u has not been visited)

- Kruskal's algorithm

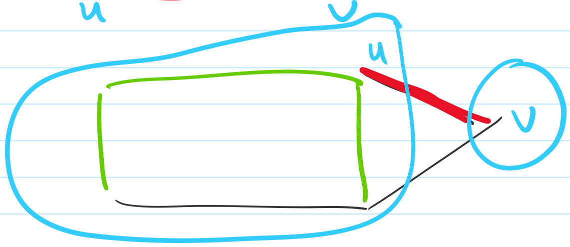








adding this edge creates a cycle
we will not add this edge



- Proof of Kruskal's algorithm.

main idea: Know general MST is correct.

if show Kruskal is a special case of general MST

then Kruskal must also be correct

going to show: every time Kruskal's algorithm adds an edge (u,v), can find a cut $(S, \bar{S})$ where $u \in S$, $v \in \bar{S}$ the cut does not contain any previous edges, and (u,v) is the min cost edge in $(S, \bar{S})$

Proof: when Kruskal adds an edge (u,v)

let S be the set of vertices that are connected to u using edges already selected by Kruskal.

by design of Kruskal, $v \notin S$

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only need to prove (u,v) is the min cost edge between $(S, \bar{S})$

assume towards contradiction that there is an edge e that crosses the cut $(S, \bar{S})$

$$w(e) < w(u,v)$$

by Kruskal, edge e is going to be considered before (u,v)

when edge e is considered, $(S, \bar{S})$ were not connected, so edge e cannot create a cycle, Kruskal must have selected edge e. this is a contradiction. $\square$

- running time

Prim: Naive $O(n^2)$

Fibonacci Heap $O(m + n\log n)$

Kruskal: $O(m\log n)$