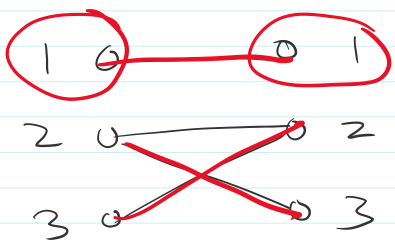
Lecture 15 Bipartite Matching

Thursday, March 7, 2019 2:40 PM

- Bipartite Graphs
  - used as abstractions for relations of two types of objects.
  - $G = (U, V, E)$ $\quad E \subseteq \{(u,v) \mid u \in U, v \in V\}$

    U: set of vertices of type 1; V: set of vertices of type 2

    every edge connects one vertex in U with another vertex in V.



$|U| = n_1 \quad |V| = n_2 \quad |M| = m$

$(1,1), (2,2), (3,2), (2,3)$

- a matching M of a bipartite graph is a subset of edges such that no two edges in M share a vertex

  $\{(1,1), (3,2), (2,3)\}$ is matching

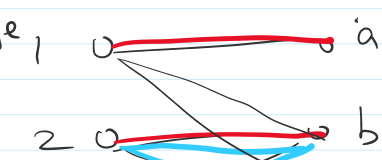
  $\{(2,2), (2,3)\}$ is not a matching

- size of a matching M is the number of edges in M
- maximum matching is a matching with maximum number of edges.

- for a bipartite graph G and a matching M
  - call a vertex "matched" if the vertex is adjacent to an edge in M

    (1, 2, a, b matched, 3, c are unmatched)
  - call an edge e matched if $e \in M$, otherwise e is unmatched
  - augmenting path: path connecting two unmatched vertices,

    the edges alternate between unmatched edge, and matched edge.

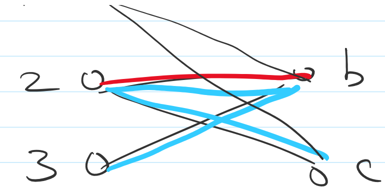


(3, b) (b, 2) (2, c) is an augmenting

and matched edge.

(3, b) (b,2) (2, c) is an augmenting path.

(3, b) (b,1) (1, c) is not an augmenting path.

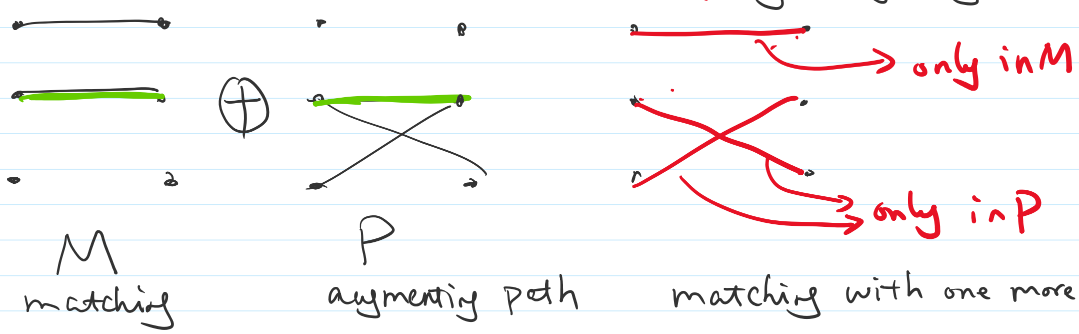


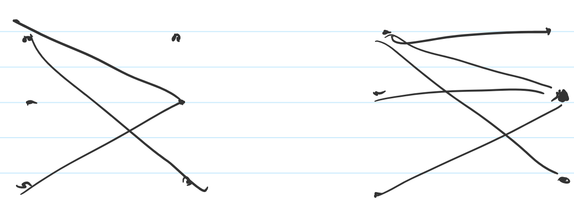
- Fact: length of augmenting path is always odd.

first and last edges of the augmenting path are unmatched.

- XOR operation $\oplus$

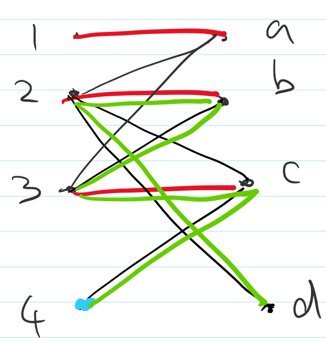
if x and y are $\{0, 1\}$ $x \oplus y$ is equal to 1 if and only if $x \neq y$ (if only one of x, y is equal to 1)

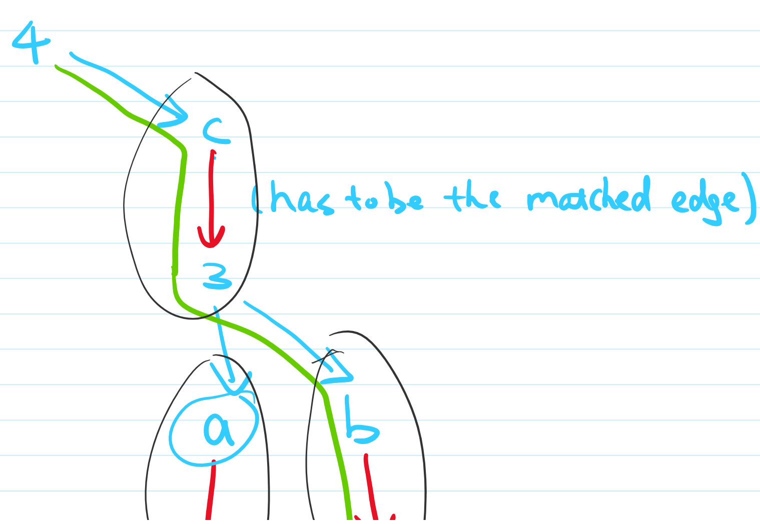


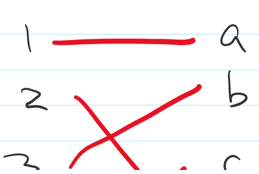


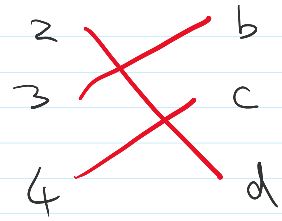
arbitrary path

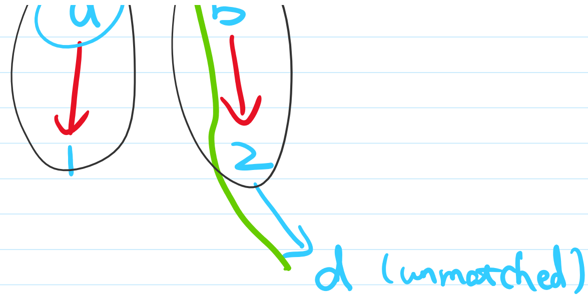
- using DFS to find augmenting path











- Proof of correctness

assume towards contradiction that M is not a maximum matching.

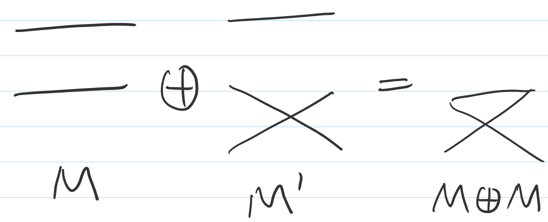
let M' be a maximum matching

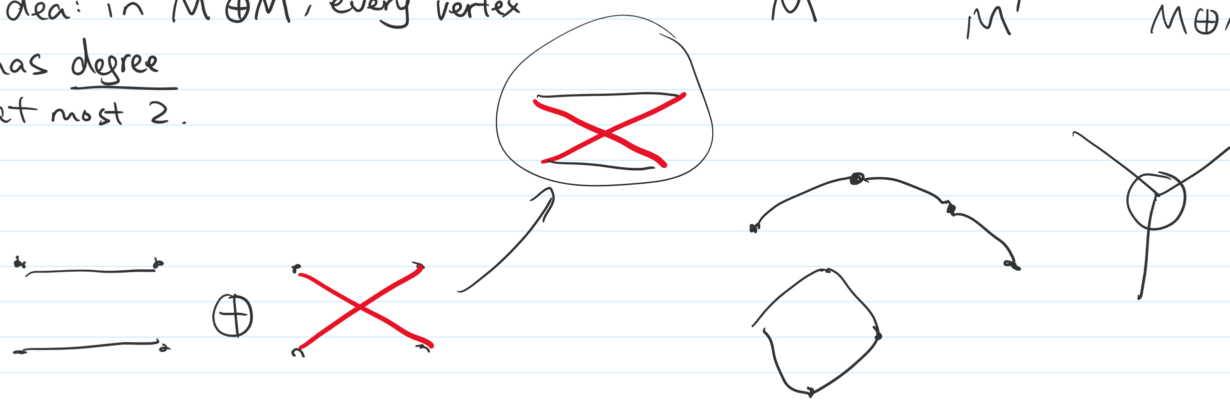
$$|M'| > |M|$$

consider $\underline{M' \oplus M}$

Claim: $\underline{M' \oplus} M$ is going to have only paths or cycles.

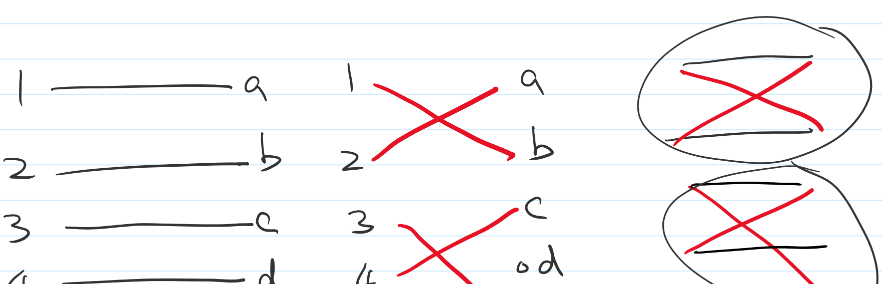
idea: in $M' \oplus M$, every vertex has $\underline{\text{degree}}$ at most 2.



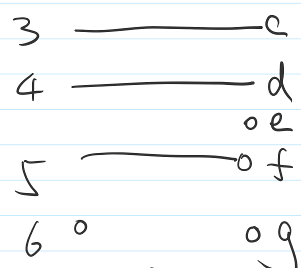


- For a cycle: it contains the same # of edges in M, M'
- For a path of odd length: one of the matching has one more edge.

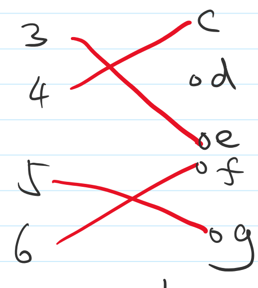
Know: $|M'| > |M|$, so there must be a path of odd length where first and last edge are M', this path is an augmenting path for M



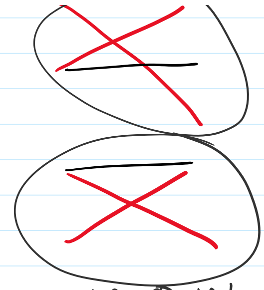
this contradicts with the assumption that M does not



M



M'



$M \oplus M'$

assumption that M does not have any augmenting path.