Lecture 16 Linear Programming

Tuesday, March 19, 2019 2:18 PM

- linear program
  - a linear program has n variables $x_1, x_2, \cdots, x_n$ (represented as a vector $x \in \mathbb{R}^n$), an objective $f(x) = c^T x = \sum_{i=1}^{n} c_i x_i$

    (x_i's are variable, c_i's are known)

    and a set of linear constraints.
  - goal of the linear program is to maximize/minimize the objective function given the constraints.
  - $$\max \; 2x_1 + x_2$$
    $$\left.\begin{array}{r} x_1 \geq 0 \\ x_2 \geq 0 \\ x_1 + x_2 \leq 1 \end{array}\right\} \longrightarrow \text{constraints.}$$

    variables: x_1 x_2

    objective $2x_1 + x_2$ $(c = (2, 1))$

- solution to a linear program
  - feasible solution: a vector x that satisfies all the constraints.

    example: $x = (0,0)$ $\begin{cases} x_1 = 0 \\ x_2 = 0 \end{cases}$ $f(x) = 0$

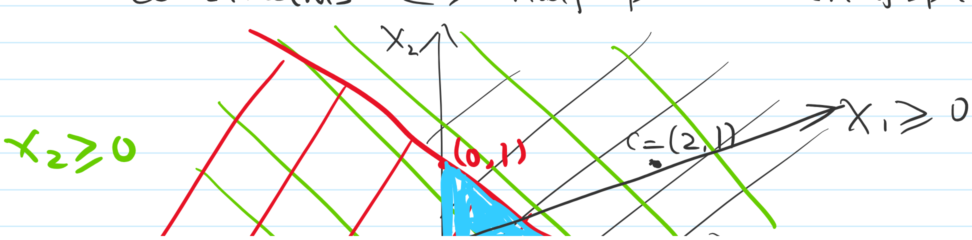
    $x = (0,1)$ $\begin{cases} x_1 = 0 \\ x_2 = 1 \end{cases}$ $f(x) = 1$

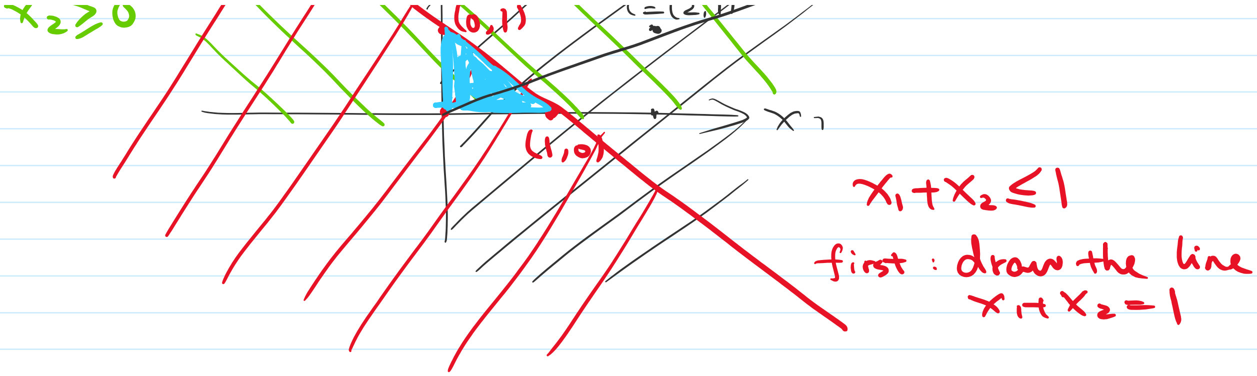
    $x = (1,1)$ $\begin{cases} x_1 = 1 \\ x_2 = 1 \end{cases}$ $x_1 + x_2 > 1$ not feasible.
  - optimal solution: a feasible solution x that optimizes the objective

    $\underline{x = (1, 0)}$ $f(x) = 2x_1 + x_2 = 2$

    value of optimal solution is the value $f(x)$ for optimal solution x

- geometric intepretation
  - constraints $\Longleftrightarrow$ half planes (half spaces in higher dimensions)





- system of linear inequalities $\Longleftrightarrow$ intersection of half planes

  blue region : feasible region

- objective function $\Longleftrightarrow$ direction of gravity

  line connecting from $\vec{O}$ to $\vec{c}$ is the direction of gravity

- canonical form

  standard way of writing linear program

  same linear program can be written in many ways.

$$\max\ 2x_1 + x_2 \qquad\qquad \min -2x_1 - x_2$$
$$\boxed{\begin{array}{c} x_1 \geq 0 \\ x_2 \geq 0 \end{array}} \qquad\qquad \begin{array}{c} x_1 \geq 0 \\ x_2 \geq 0 \end{array}$$
$$x_1 + x_2 \leq 1 \qquad\qquad 1 - x_1 - x_2 \geq 0$$

- canonical form

$$\min\ \langle c, x \rangle = c^T x = \sum_{i=1}^{n} c_i x_i$$

m inequalities $Ax \geq b \Longleftrightarrow$ for every $j = 1, 2, \cdots, m$ $\quad \sum_{i=1}^{n} A_{j,i} x_i \geq b_j$

n inequalities $x \geq 0 \Longleftrightarrow$ for every $i = 1, 2, \cdots, n$ $\quad x_i \geq 0$

$$m \underbrace{\boxed{\quad A \quad}}_{n} \ \boxed{x}\ n = \boxed{\ }\ m$$

for $j = 1, 2, \ldots, m$
$$(Ax)_j = \sum_{i=1}^{n} A_{j,i} x_i$$

for the example LP $\quad c = (-2, -1)$

for the example LP $c = (-2, -1)$

$A = (-1 \quad -1) \qquad b = -1$

$Ax \geq b \iff -x_1 - x_2 \geq -1$

- free variable

$$\max\ x + y$$
$$y \geq 0$$
$$x + 2y \leq 3$$
$$x \geq -5$$

call x a free variable because we have no constraint $x \geq 0$

$x = x_1 - x_2$
$x_1 \geq 0$
$x_2 \geq 0$

$$\max\ x_1 - x_2 + y$$
$$y \geq 0$$
$$x_1 \geq 0$$
$$x_2 \geq 0$$
$$x_1 - x_2 + 2y \leq 3$$
$$x_1 - x_2 \geq -5 \quad ②$$

$$-x_1 + x_2 - 2y \geq -3 \quad ①$$

$$\begin{array}{c} ① \\ ② \end{array} \underbrace{\begin{pmatrix} -1 & 1 & -2 \\ 1 & -1 & 0 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \\ y \end{pmatrix} = \underbrace{\begin{pmatrix} -3 \\ -5 \end{pmatrix}}_{b}$$

- using LP to solve graph problems
  - Example 1: bipartite matching

① find the variables

in matching: for each edge want to know if it's in the matching.

$$x_{i,j} = \begin{cases} 1 & \text{if } (i,j) \text{ is an edge in matching} \\ 0 & \text{if } (i,j) \text{ is not in matching} \end{cases}$$

② write constraints.

a. every classroom can hold ≤ 1 course

for every j (classroom) $\left(\sum_{i=1}^{n} x_{i,j}\right) \leq 1$

for every j (classroom) $\sum_{i=1}^{n} x_{ij} \le 1$

b. every course needs only ≤ 1 classroom

for every i (course) $\sum_{j=1}^{n} x_{ij} \le 1$

c. x_{ij} should be 0 or 1

cannot write this constraint, try best approximation.

$0 \le x_{ij} \le 1$ (possible $x_{ij} = 0.5$)

③ write the objective

maximize $\underbrace{\text{number of matched edges}}$

$$\sum_{(i,j)\in E} x_{ij}$$

$$\max \boxed{\sum_{(i,j)\in E} x_{ij}}$$

for bipartite matching

option ① have x_{ij} variables for all (i,j) pairs

only place where graph influences the LP is in the objective

② only have x_{ij} variables if (i,j) is an edge.

for every i $\sum_{j:(i,j)\in E} x_{ij} \le 1$

- fractional solution vs. integral solution

for LP, variables can be fractions in general.

if all variables are integers, call that an integral solution

usually we hope for integral solutions

- for the matching LP, possible to prove there exists integral optimal solution