Lecture 19 Randomized Algorithms

Thursday, April 4, 2019 2:49 PM

- random variable: X is a random variable if it's value depends on some random events.

  $\Pr[X = v]$ denotes the probability that X is equal to v

  e.g. if $X \sim$ coin $\quad X = \begin{cases} 0 & \text{w.p. } \frac{1}{2} \\ 1 & \text{w.p. } \frac{1}{2} \end{cases}$

  $X = \begin{cases} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{cases} \quad \text{w.p. } \frac{1}{6}$

- $X = v \quad X > v \quad X < v$ are called "events"

  $X \in S$

- joint probability: probability that two events happen at the same time.

  $\Pr[X = i, Y = j]$

  independence if X, Y are independent.

  $$\Pr[X = i, Y = j] = \Pr[X = i] \Pr[Y = j]$$

- conditional probability

  $\Pr[X = i \mid \underbrace{Y = j}_{\text{condition}}]$: probability of $X = i$, given that we already know $Y = j$

  if X, Y are independent

  $$\Pr[X = i \mid Y = j] = \Pr[X = i]$$

  Bayes rule

  $$\Pr[X = i \mid Y = j] = \frac{\Pr[X = i, Y = j]}{\Pr[Y = j]}$$

  $$= \frac{\Pr[Y = j \mid X = i] \cdot \Pr[X = i]}{\Pr[Y = j]}$$

- expectation

  $E[X] = \sum \Pr[X = i] \cdot i$

- expectation

$$\mathbb{E}[X] = \sum_i \Pr[X=i]\, x_i$$

linearity of expectation

$$\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$$

true even if X and Y are not independent.

$$\mathbb{E}[3X+2Y+Z] = 3\mathbb{E}[X] + 2\mathbb{E}[Y] + \mathbb{E}[Z]$$

- conditional expectation

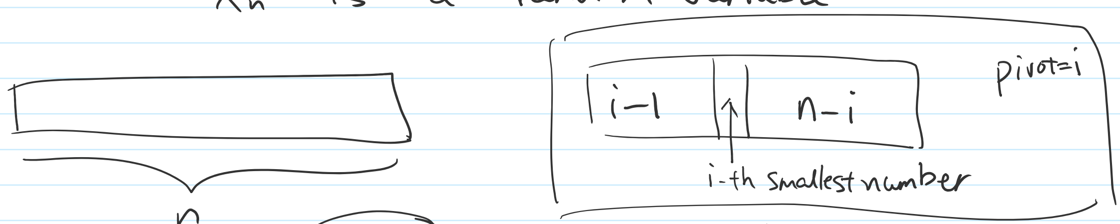
$$\mathbb{E}[X \mid Y=j] = \sum_i \Pr[X=i \mid Y=j]\, x_i$$

$$\mathbb{E}[X] = \sum_j \mathbb{E}[X \mid Y=j] \times \Pr[Y=j]$$

$$= \mathbb{E}_Y\left[\mathbb{E}[X \mid Y]\right]$$

- Quick Sort
  - Let X_n be running time of quicksort on arrey of length n

    X_n is a random veriable



pivot = i event that the first pivot number is equal to the i-th smallest number in the array (for simplicity, assume array has distinct elements)

$$\mathbb{E}[X_n] = \sum_{i=1}^{n} \underline{\mathbb{E}[X_n \mid \text{pivot}=i]} \times \underline{\Pr[\text{pivot}=i]}$$

for any i $\quad \Pr[\text{pivot}=i] = \frac{1}{n}$

$$\mathbb{E}[X_n \mid \text{pivot}=i] = \mathbb{E}[X_{i-1} + \underbrace{A \cdot n}_{} + X_{n-i} \mid \text{pivot}=i]$$

time takes to sort   time takes to split the arrey   time takes to sort the time takes to sort the left part

time takes to split the array

time takes to sort the right part

$$= \mathbb{E}[X_{i-1}] + An + \mathbb{E}[X_{n-i}]$$

$$\mathbb{E}[X_n] = \frac{1}{n}\sum_{i=1}^{n}\left(\mathbb{E}[X_{i-1}] + \mathbb{E}[X_{n-i}] + A\cdot n\right)$$

induction hypothesis

$\mathbb{E}[X_n] \leq C\cdot n\log_2 n$ (C is chosen later)

base case $\mathbb{E}[X_1] = 0$ ✓

induction: assume $\mathbb{E}[X_k] \leq \boxed{C\cdot k\cdot\log_2 k}$ for all $k<n$

$$\mathbb{E}[X_n] = \frac{1}{n}\sum_{i=1}^{n}\left(\mathbb{E}[X_{i-1}] + \mathbb{E}[X_{n-i}] + \underline{A\cdot n}\right)$$

$$= A\cdot n + \frac{1}{n}\sum_{i=1}^{n}\left(\mathbb{E}[X_{i-1}] + \mathbb{E}[X_{n-i}]\right)$$

$$\mathbb{E}[X_0] + \mathbb{E}[X_1] + \cdots + \mathbb{E}[X_{n-1}] \qquad \mathbb{E}[X_{n-1}] + \mathbb{E}[X_{n-2}] + \cdots + \mathbb{E}[X_0]$$

$$= A\cdot n + 2\times\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}[X_{i-1}]$$

$$\leq A\cdot n + 2\times\frac{1}{n}\sum_{i=2}^{n} C\cdot(i-1)\cdot\log_2(i-1)$$

$$\boxed{\log_2(i-1) \leq \log_2\frac{n}{2} \text{ if } i\leq\frac{n}{2}}$$

$\log_2(i-1) \leq \log_2 n$ for all $i\leq n$

$$\leq A\cdot n + 2\times\frac{1}{n}\left(\sum_{i=2}^{\frac{n}{2}} C\cdot(i-1)\cdot\log_2\frac{n}{2} + \sum_{i=\frac{n}{2}+1}^{n} C\cdot(i-1)\cdot\log_2 n\right)$$

$$= A\cdot n + 2\times\frac{1}{n}\left(\sum_{i=2}^{n} C\cdot(i-1)\cdot\log_2 n - \sum_{i=2}^{\frac{n}{2}} C\cdot(i-1)\right)$$

$$\log_2\frac{n}{2} = \log_2 n - 1$$

$$= \underline{A\cdot n + C\cdot n\log_2 n - C\cdot\frac{n}{4}}$$

$$\leq C\cdot n\log_2 n$$

true when $C \geq 4A$

$\leq C \cdot n \log_2 n$

true when $C \geq 4 \cdot A$

by induction can choose $C = 4A$

$E[X_n] \leq 4A\, n \log_2 n$