COMPSCI 330: Design and Analysis of Algorithms **Spring 2019**

Lecture 3: Divide and Conquer

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3.1 Integer multiplication

Input: Two positive n digit integers, a and b.
Output: $a * b$
The naive approach is to simply multiply the two numbers which takes $O(n^2)$ time because we need to multiply each digit in one number with those in the second number, put them in the correct position and add them as shown below.

$$\begin{array}{r} 3\ 8\ 4 \\ \times \quad 5\ 6 \\ \hline 2\ 3\ 0\ 4 \\ 1\ 9\ 2\ 0\ \\ \hline 2\ 1\ 5\ 0\ 4 \end{array}$$

Can we do better?

Suppose we are given $a = 123456$ and $b = 654321$. We can rewrite a as $123000 + 456$ and b as $654000 + 321$. As a result, $a * b = 123 * 654 * 10^6 + (123 * 321 + 456 * 654) * 10^3 + 456 * 321$.

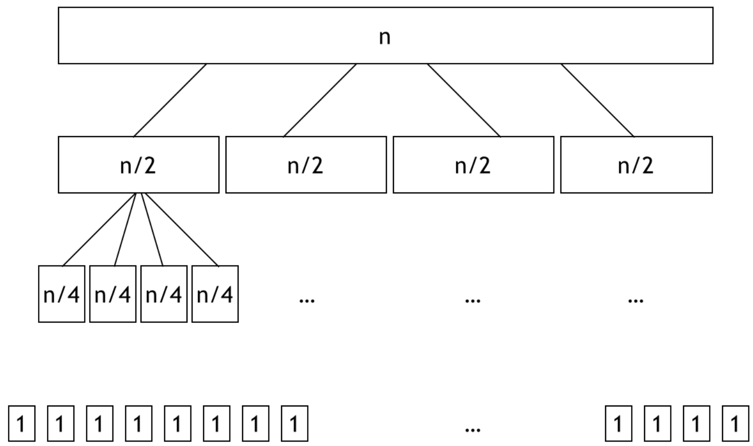
Assume n is a power of 2. We partition a and b into lower and upper digits as $a = a1 * 10^{n/2} + a_2$ and $b = b_1 * 10^{n/2} + b_2$. Thus, the product becomes $A * 10^n + (B + C) * 10^3 + D$, where $A = a1 * b1, B = a2 * b1, C = a1 * b2$ and $D = a2 * b2$.

Recursion: Let $T(n)$ be the running time to multiply two n-digit numbers, a and b.
$Multiply(a, b)$:

1. WLOG assume $n = length(a) = length(b)$, can pad 0's for shorter number
2. if $length(a) <= 1$ then return $a * b$
3. Partition a,b into $a = a1 * 10^{n/2} + a2$ and $b = b1 * 10^{n/2} + b2$
4. $A = Multiply(a1, b1)$
5. $B = Multiply(a2, b1)$
6. $C = Multiply(a1, b2)$
7. $D = Multiply(a2, b2)$
8. Return $A * 10^n + (B + C) * 10^{n/2} + D$

Thus,

$$T(n) = 4T(\frac{n}{2}) + O(n)$$



where $O(n)$ accounts for partitioning the given numbers, the addition operation(s) and shifting/padding. We call this the 'merge' time.

$$\begin{aligned} T(n) &= 4T(n/2) + A * n && \text{(A*n indicates the merging cost at layer 0)} \\ &= 16T(n/4) + 4 * A * n/2 + A * n \\ &= 64T(n/8) + 16 * A * n/4 + 4 * A * n/2 + A * n \end{aligned}$$

We can assume $T(1) = 1$ since we are doing asymptotic analysis. Overall cost of the function is the sum of the merging cost of all layers. The number of layers $l = \log_2 n$.

$$\begin{aligned} T(n) &= \sum_{i=0}^{\log_2 n} 4^i A \frac{n}{2^i} \\ &= An \sum_{i=0}^{\log_2 n} 2^i \\ &= An(2n-1) = O(n^2) \end{aligned}$$

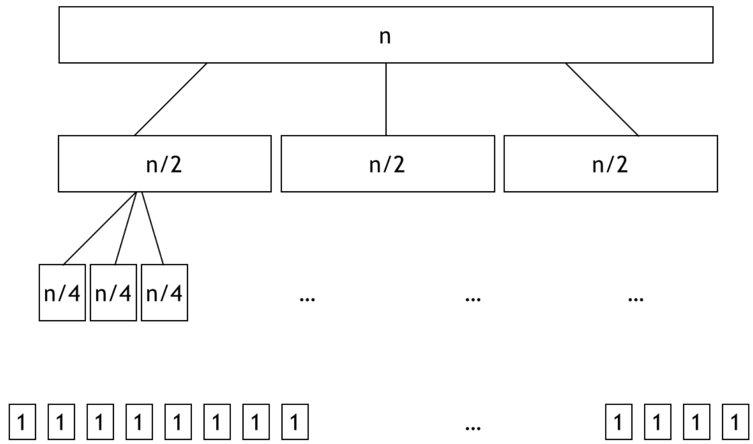
3.2 Improved algorithm

We can improve the algorithm by doing one of the following:

1. Merging faster: However, this is not the bottleneck for integer multiplication. $O(n)$ is not large.
2. Make subproblems smaller: If we do this naively, then that would result in more number of subproblems which defeats the purpose.
3. Decrease the number of subproblem: We see the details below.

$Multiply(a, b)$:

1. WLOG assume $n = length(a) = length(b)$, can pad 0's for shorter number



2. if $length(a) <= 1$ then return $a * b$
3. Partition a,b into $a = a1 * 10^{n/2} + a2$ and $b = b1 * 10^{n/2} + b2$
4. $A = Multiply(a1, b1)$
5. $B = Multiply(a2, b2)$
6. $C = Multiply(a1 + a1, b1 + b2)$
7. Return $A * 10^n + (C - A - B) * 10^{n/2} + B$

Thus,

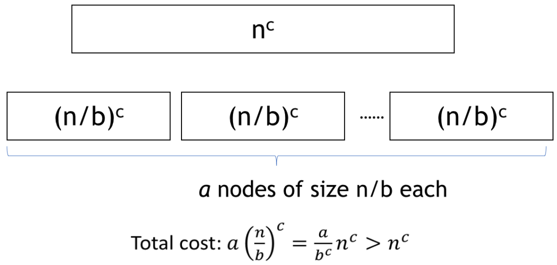
$$T(n) = 3T(\frac{n}{2}) + O(n)$$

$$\begin{aligned}
T(n) &= \sum_{i=0}^{\log_2 n} 3^i A \frac{n}{2^i} \\
&= An \sum_{i=0}^{\log_2 n} (\frac{3}{2})^i \\
&= An \frac{(3/2)^{\log_2 n+1} - 1}{3/2 - 1} \\
&= O(n \frac{3}{2}^{\log_2 n}) \\
&= O(n * n^{\log_2 3/2}) \\
&= O(n^{\log_2 3}) \\
&= O(n^{1.585})
\end{aligned}$$

Even faster algorithms use fast Fourier analysis, which is beyond the scope of this class.

3.3 Master theorem

The Master Theorem acts as a "cheat sheet" for basic recursions. Given $T(n) = aT(\frac{n}{b}) + f(n)$):



1. If $f(n) = O(n^c), c < \log_b(a)$ then $T(n) = \Theta(n^{\log_b a})$
2. If $f(n) = \Theta(n^c \log^t(n)), c = \log_b a$ then $T(n) = \Theta(n^{\log_b a} \log^{t+1}(n))$
3. If $f(n) = \Theta(n^c), c > log_b a$ then $T(n) = \Theta(n^c)$

Case 1: Merge cost is dominated by the cost of the last layer.

$l :=$ number of layers, and equals $\log_b n$.
number of nodes in layer l equals a^l, and the merge cost for this last layer equals $n^{\log_b a}$.
Example. When $a = 4, b = 2$ and $f(n) = n$: as seen for the first algorithm for integer multiplication, we get $O(n^{\log_2 4})$.
When $a = 3, b = 2$ and $f(n) = n$: as seen for the second algorithm for integer multiplication, we get $O(n^{\log_2 3})$.

Case 2: Addtional log factor shows up in the overall runtime because of the height of the recursion tree, i.e., the number of layers.
Example. $a = 2, b = 2$ and $f(n) = n$: applies to mergesort and counting inversions which we saw in a previous lecture. Here, we get $O(n \log n)$.

Case 3: Merge cost is dominated by the cost of the first layer.

Cannot improve further by reducing a. Can instead try to improve the merge cost from something smaller than n^c.
Example. Running time for the first attempt to counting the number of inversions. There, we had $a = 2, b = 2$ and $f(n) = n^2$, which gave an overall runtime of $O(n^2)$.