Mathematical Induction

Lecture 10-11

Menu

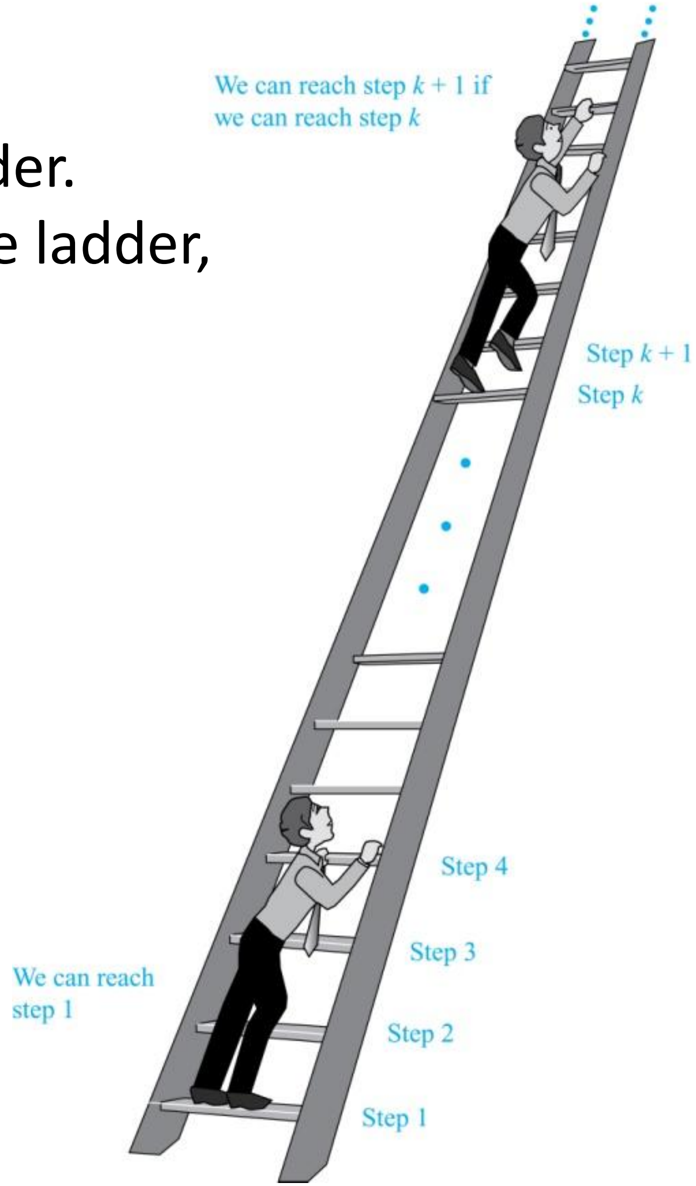
- Mathematical Induction
- Strong Induction
- Recursive Definitions
- Structural Induction

Climbing an Infinite Ladder

Suppose we have an infinite ladder:

1. We can reach the first rung of the ladder.
2. If we can reach a particular rung of the ladder, then we can reach the next rung.

From (1), we can reach the first rung. Then by applying (2), we can reach the second rung. Applying (2) again, the third rung. And so on. We can apply (2) any number of times to reach any particular rung, no matter how high up.



This example motivates proof by mathematical induction.

Principle of Mathematical Induction

Principle of Mathematical Induction: To prove that $P(n)$ is true for all positive integers n, we complete these steps:

Basis Step: Show that $P(1)$ is true.

Inductive Step: Show that $P(k) \rightarrow P(k+1)$ is true for all positive integers k.

To complete the inductive step, assuming the inductive hypothesis that $P(k)$ holds for an arbitrary integer k, show that must $P(k+1)$ be true.

Climbing an Infinite Ladder Example:

Basis Step: By (1), we can reach rung 1.

Inductive Step: Assume the inductive hypothesis that we can reach rung k. Then by (2), we can reach rung $k+1$.

Hence, $P(k) \rightarrow P(k+1)$ is true for all positive integers k. We can reach every rung on the ladder.

Logic and Mathematical Induction

- Mathematical induction can be expressed as the rule of inference

$$(P(1) \land \forall k\,(P(k) \to P(k+1))) \to \forall n\, P(n),$$

where the domain is the set of positive integers.

- In a proof by mathematical induction, we don't assume that $P(k)$ is true for all positive integers! We show that if we assume that $P(k)$ is true, then $P(k + 1)$ must also be true.
- Proofs by mathematical induction do not always start at the integer 1. In such a case, the basis step begins at a starting point b where b is an integer. We will see examples of this soon.

Why Mathematical Induction is Valid?

- Mathematical induction is valid because of the well ordering property.
- Proof:
  - Suppose that $P(1)$ holds and $P(k) \rightarrow P(k+1)$ is true for all positive integers k.
  - Assume there is at least one positive integer n for which $P(n)$ is false. Then the set S of positive integers for which $P(n)$ is false is nonempty.
  - By the well-ordering property, S has a least element, say m.
  - We know that m can not be 1 since $P(1)$ holds.
  - Since m is positive and greater than 1, $m - 1$ must be a positive integer. Since $m - 1 < m$, it is not in S, so $P(m - 1)$ must be true.
  - But then, since the conditional $P(k) \rightarrow P(k + 1)$ for every positive integer k holds, $P(m)$ must also be true. This contradicts $P(m)$ being false.
  - Hence, $P(n)$ must be true for every positive integer n.

Proving a Summation Formula by Mathematical Induction

Example: Show that: $\sum_{i=1}^{n} = \frac{n(n+1)}{2}$

Solution:

- BASIS STEP: *P*(1) is true since 1(1 + 1)/2 = 1.
- INDUCTIVE STEP: Assume true for *P*(*k*).

  The inductive hypothesis is $\sum_{i=1}^{k} = \frac{k(k+1)}{2}$

  Under this assumption,

$$\begin{aligned} 1 + 2 + \ldots + k + (k+1) &= \frac{k(k+1)}{2} + (k+1) \\ &= \frac{k(k+1) + 2(k+1)}{2} \\ &= \frac{(k+1)(k+2)}{2} \end{aligned}$$

Exercise:

- Show that the sum of first n positive odd numbers is n^2.
- We will do it on the board!

Proving Inequalities

Example: Use mathematical induction to prove that $n < 2^n$ for all positive integers n.

Solution: Let $P(n)$ be the proposition that $n < 2^n$.

- Basis Step: (1) is true since $1 < 2^1 = 2$.
- Inductive Step: Assume $P(k)$ holds, i.e., $k < 2^k$, for an arbitrary positive integer k.
- Must show that $P(k + 1)$ holds. Since by the inductive hypothesis, $k < 2^k$, it follows that:

$$k + 1 < 2^k + 1 \leq 2^k + 2^k = 2 \cdot 2^k = 2^{k+1}$$

Therefore $n < 2^n$ holds for all positive integers n. ◀

Proving Inequalities

Example: Use mathematical induction to prove that $2^n < n!$, for every integer $n \geq 4$.

Solution: Let $P(n)$ be the proposition that $2^n < n!$.

- Basis: $P(4)$ is true since $2^4 = 16 < 4! = 24$.
- Inductive Step: Assume $P(k)$ holds, i.e., $2^k < k!$ for an arbitrary integer $k \geq 4$. To show that $P(k + 1)$ holds:

$$\begin{aligned} 2^{k+1} &= 2 \cdot 2^k \\ &< 2 \cdot k! \qquad \textit{(by the inductive hypothesis)} \\ &< (k + 1)k! \\ &= (k + 1)! \end{aligned}$$

Therefore, $2^n < n!$ holds, for every integer $n \geq 4$. ◀

Note: The basis step is $P(4)$, since $P(0)$, $P(1)$, $P(2)$, and $P(3)$ are all false.

Example

Example: Use mathematical induction to prove that $n^3 - n$ is divisible by 3, for every positive integer n.

Solution: Let $P(n)$ be the proposition that $3 \mid (n^3 - n)$.

- Basis: $P(1)$ is true since $1^3 - 1 = 0$, which is divisible by 3.
- Induction: Assume $P(k)$ holds, i.e., $k^3 - k$ is divisible by 3, for an arbitrary positive integer k. To show that $P(k + 1)$ follows:

$$(k+1)^3 - (k+1) = (k^3 + 3k^2 + 3k + 1) - (k+1)$$
$$= (k^3 - k) + 3(k^2 + k)$$

  By the inductive hypothesis, the first term $(k^3 - k)$ is divisible by 3 and the second term is divisible by 3 since it is an integer multiplied by 3. So by part (i) of Theorem 1 in Section 4.1 , $(k+1)^3 - (k+1)$ is divisible by 3.

Therefore, $n^3 - n$ is divisible by 3, for every integer positive integer n. ◀

Strong Induction

Strong Induction: To prove that $P(n)$ is true for all positive integers n, where $P(n)$ is a propositional function, complete two steps:

Basis Step: Verify that the proposition $P(1)$ is true.

Inductive Step: Show the conditional statement $[P(1) \wedge P(2) \wedge \cdots \wedge P(k)] \rightarrow P(k+1)$ holds for all positive integers k.

Strong Induction is sometimes called the *second principle of mathematical induction* or *complete induction*.

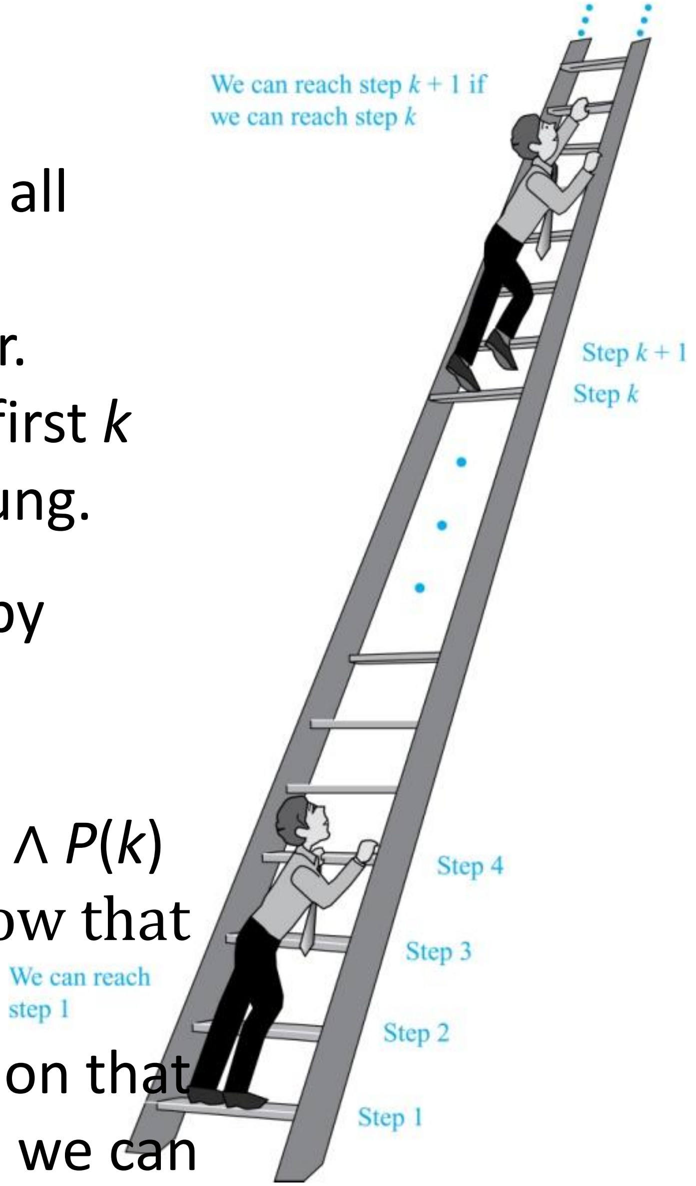
Strong induction tells us that we can reach all rungs if:

1. We can reach the first rung of the ladder.
2. For every integer k, if we can reach the first k rungs, then we can reach the $(k + 1)$st rung.

To conclude that we can reach every rung by strong induction:

- BASIS STEP: $P(1)$ holds
- INDUCTIVE STEP: Assume $P(1) \wedge P(2) \wedge \cdots \wedge P(k)$ holds for an arbitrary integer k, and show that $P(k + 1)$ must also hold.

We will have then shown by strong induction that for every positive integer n, $P(n)$ holds, i.e., we can reach the nth rung of the ladder.



Proof using Strong Induction

Example: Suppose we can reach the first and second rungs of an infinite ladder, and we know that if we can reach a rung, then we can reach two rungs higher. Prove that we can reach every rung.

Solution: Prove the result using strong induction.

- BASIS STEP: We can reach the first step.
- INDUCTIVE STEP: The inductive hypothesis is that we can reach the first *k* rungs, for any $k \geq 2$. We can reach the $(k+1)$st rung since we can reach the $(k-1)$st rung by the inductive hypothesis.

Hence, we can reach all rungs of the ladder. ◀

Strong vs Mathematical Induction

- We can always use strong induction instead of mathematical induction. But there is no reason to use it if it is simpler to use mathematical induction.
- In fact, the principles of mathematical induction, strong induction, and the well-ordering property are all equivalent.
- Sometimes it is clear how to proceed using one of the three methods, but not the other two.

Example

Example: Show that if n is an integer greater than 1, then n can be written as the product of primes.

Solution: Let $P(n)$ be the proposition that n can be written as a product of primes.

- BASIS STEP: $P(2)$ is true since 2 itself is prime.
- INDUCTIVE STEP: The inductive hypothesis is $P(j)$ is true for all integers j with $2 \leq j \leq k$. To show that $P(k + 1)$ must be true under this assumption, two cases need to be considered:
  - If $k + 1$ is prime, then $P(k + 1)$ is true.
  - Otherwise, $k + 1$ is composite and can be written as the product of two positive integers a and b with $2 \leq a \leq b < k + 1$. By the inductive hypothesis a and b can be written as the product of primes and therefore $k + 1$ can also be written as the product of those primes.

Hence, it has been shown that every integer greater than 1 can be written as the product of primes. ◀

Proof using Strong Induction

Example: Prove that every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps.

Solution: Let $P(n)$ be the proposition that postage of n cents can be formed using 4-cent and 5-cent stamps.

- BASIS STEP: $P(12)$, $P(13)$, $P(14)$, and $P(15)$ hold.
  - $P(12)$ uses three 4-cent stamps.
  - $P(13)$ uses two 4-cent stamps and one 5-cent stamp.
  - $P(14)$ uses one 4-cent stamp and two 5-cent stamps.
  - $P(15)$ uses three 5-cent stamps.
- INDUCTIVE STEP: The inductive hypothesis states that $P(j)$ holds for $12 \leq j \leq k$, where $k \geq 15$. Assuming the inductive hypothesis, it can be shown that $P(k + 1)$ holds.
- Using the inductive hypothesis, $P(k - 3)$ holds since $k - 3 \geq 12$. To form postage of $k + 1$ cents, add a 4-cent stamp to the postage for $k - 3$ cents.

Hence, $P(n)$ holds for all $n \geq 12$. ◀

Recursive Definitions
and
Structural Induction

Recursively Defined Functions

Definition: A recursive or inductive definition of a function consists of two steps.

- BASIS STEP: Specify the value of the function at zero.
- RECURSIVE STEP: Give a rule for finding its value at an integer from its values at smaller integers.

- A function $f(n)$ is the same as a sequence a_0, a_1, … , where a_i, where $f(i) = a_i$.

Example: Fibonacci Sequence

Define the Fibonacci sequence, $f_0, f_1, f_2, \ldots$, by:

- Initial Conditions: $f_0 = 0, f_1 = 1$
- Recurrence Relation: $f_n = f_{n-1} + f_{n-2}$

f_{-6}	f_{-5}	f_{-4}	f_{-3}	f_{-2}	f_{-1}	f_0	f_1	f_2	f_3	f_4	f_5	f_6
-8	**5**	**-3**	**2**	**-1**	**1**	**0**	**1**	**1**	**2**	**3**	**5**	**8**

Recursively Defined Sets and Structures

Recursive definitions of sets have two parts:

- The *basis step* specifies an initial collection of elements.
- The *recursive step* gives the rules for forming new elements in the set from those already known to be in the set.

- Sometimes the recursive definition has an *exclusion rule*, which specifies that the set contains nothing other than those elements specified in the basis step and generated by applications of the rules in the recursive step.
- We will always assume that the exclusion rule holds, even if it is not explicitly mentioned.
- We will later develop a form of induction, called *structural induction*, to prove results about recursively defined sets.

Examples

- Factorial of n
  - n! = 1 if n = 0
  - n! = n (n-1)!, otherwise
- Sum of first n odd numbers S_n
  - S_n = 1 if n = 1
  - $S_n = S_{n-1} + (2n - 1)$, otherwise
- Length of a string s ∊ Σ^*: len(s)
  - len(*s*) = 0 if s = ε
  - len(*sa*) = len(*s*) + 1 if *s* ∊ Σ* and *a* ∊ Σ*.*
- Sorting n numbers: SORT(<a1, …, an>)
  - <a1> = SORT(<a1, …, an>) if n = 1
  - SORT(<a1, …, an>) = <min(<a1, …, an>),
    SORT <a1, …, an> - < min(<a1, …, an>)>)

String Concatenation

Definition: Two strings can be combined via the operation of *concatenation*. Let Σ be a set of symbols and Σ* be the set of strings formed from the symbols in Σ. We can define the concatenation of two strings, denoted by ·, recursively as follows.

BASIS STEP: If $w \in \Sigma^*$, then $w \cdot \varepsilon = w$.

RECURSIVE STEP: If $w_1 \in \Sigma^*$ and $w_2 \in \Sigma^*$ and $x \in \Sigma$, then $w \cdot (w_2 x) = (w_1 \cdot w_2)x$.

- Often $w_1 \cdot w_2$ is written as $w_1 w_2$.
- If $w_1 = abra$ and $w_2 = cadabra$, the concatenation $w_1 w_2 = abracadabra$.

Balanced Parentheses

Example: Give a recursive definition of the set of balanced parentheses P.

Solution:

BASIS STEP: $() \in P$

RECURSIVE STEP: If $w \in P$, then $()w \in P$, $(w) \in P$ and $w() \in P$.

- Show that $(() ())$ is in P.
- Why is $))(()$ not in P?

Well-Formed Formulae in Propositional Logic

Definition: The set of *well-formed formulae* in propositional logic involving **T**, **F**, propositional variables, and operators from the set $\{\neg, \wedge, \vee, \rightarrow, \leftrightarrow\}$.

BASIS STEP: **T**,**F**, and *s*, where *s* is a propositional variable, are well-formed formulae.

RECURSIVE STEP: If E and F are well formed formulae, then $(\neg E)$, $(E \wedge F)$, $(E \vee F)$, $(E \rightarrow F)$, $(E \leftrightarrow F)$, are well-formed formulae.

Examples: $((p \vee q) \rightarrow (q \wedge \mathbf{F}))$ is a well-formed formula.

$pq \wedge$ is not a well formed formula.

Structural Induction

Definition: To prove a property of the elements of a recursively defined set, we use *structural induction*.

BASIS STEP: Show that the result holds for all elements specified in the basis step of the recursive definition.

RECURSIVE STEP: Show that if the statement is true for each of the elements used to construct new elements in the recursive step of the definition, the result holds for these new elements.

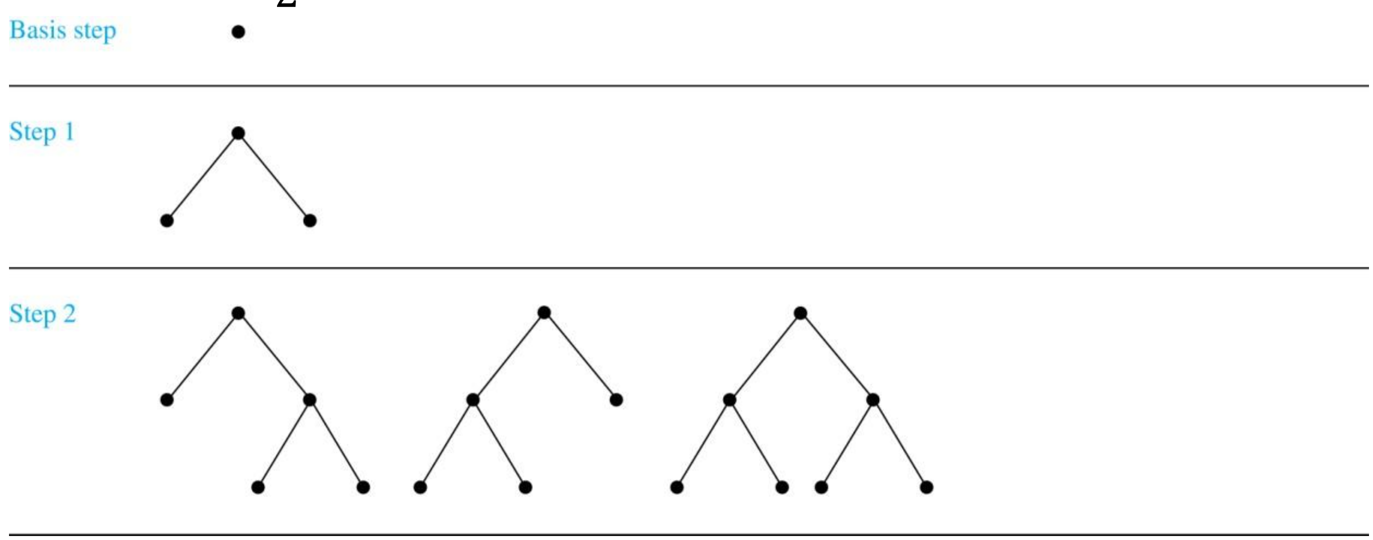
- The validity of structural induction can be shown to follow from the principle of mathematical induction.

Example: Full Binary Trees

Definition: The set of *full binary trees* can be defined recursively by these steps.

BASIS STEP: There is a full binary tree consisting of only a single vertex *r*.

RECURSIVE STEP: If T_1 and T_2 are disjoint full binary trees, there is a full binary tree, denoted by $T_1 \cdot T_2$, consisting of a root *r* together with edges connecting the root to each of the roots of the left subtree T_1 and the right subtree T_2.



Example: Full Binary Trees

Definition: The *height* $h(T)$ of a full binary tree T is defined recursively as follows:

- BASIS STEP: The height of a full binary tree T consisting of only a root r is $h(T) = 0$.
- RECURSIVE STEP: If T_1 and T_2 are full binary trees, then the full binary tree $T = T_1 \cdot T_2$ has height
  $h(T) = 1 + \max(h(T_1), h(T_2))$.

- The number of vertices $n(T)$ of a full binary tree T satisfies the following recursive formula:
  - BASIS STEP: The number of vertices of a full binary tree T consisting of only a root r is $n(T) = 1$.
  - RECURSIVE STEP: If T_1 and T_2 are full binary trees, then the full binary tree $T = T_1 \cdot T_2$ has the number of vertices
    $$n(T) = 1 + n(T_1) + n(T_2).$$

Example: Full Binary Trees

Theorem: If T is a full binary tree, then $n(T) \leq 2^{h(T)+1} - 1.$

Proof: Use structural induction.

- BASIS STEP: The result holds for a full binary tree consisting only of a root, $n(T) = 1$ and $h(T) = 0$. Hence, $n(T) = 1 \leq 2^{0+1} - 1 = 1$.
- RECURSIVE STEP: Assume $n(T_1) \leq 2^{h(T1)+1} - 1$ and also $n(T_2) \leq 2^{h(T2)+1} - 1$ whenever T_1 and T_2 are full binary trees.

$n(T) = 1 + n(T_1) + n(T_2)$ (*by recursive formula of n(T)*)

$\leq 1 + (2^{h(T1)+1} - 1) + (2^{h(T2)+1} - 1)$ (*by inductive hypothesis*)

$\leq 2 \cdot \max(2^{h(T1)+1}, 2^{h(T2)+1}) - 1$

$= 2 \cdot 2^{\max(h(T1),h(T2))+1} - 1$ ($\max(2^x, 2^y) = 2^{\max(x,y)}$)

$= 2 \cdot 2^{h(t)} - 1$ (*by recursive definition of h(T)*)

$= 2^{h(t)+1} - 1$

Generalized Induction

- *Generalized induction* is used to prove results about sets other than the integers that have the well-ordering property.
- For example, consider an ordering on **N**$\times$ **N**, ordered pairs of nonnegative integers. Specify that (x_1, y_1) is less than or equal to (x_2, y_2) if either $x_1 < x_2$, or $x_1 = x_2$ and $y_1 < y_2$. This is called the *lexicographic ordering*.
- Strings are also commonly ordered by a *lexicographic ordering*.
- The next example uses generalized induction to prove a result about ordered pairs from **N**$\times$ **N**.

Generalized Induction

Example: Suppose that $a_{m,n}$ is defined for $(m,n) \in \mathbf{N} \times \mathbf{N}$ by $a_{0,0} = 0$ and

$$a_{m,n} = \begin{cases} a_{m-1,n} + 1 & \text{if } n = 0 \text{ and } m > 0 \\ a_{m,n-1} + n & \text{if } n > 0 \end{cases}.$$

Show that $a_{m,n} = m + n(n+1)/2$ is defined for all $(m,n) \in \mathbf{N} \times \mathbf{N}$.

Solution: Use generalized induction.

BASIS STEP: $a_{0,0} = 0 = 0 + (0 \cdot 1)/2$

INDUCTIVE STEP: Assume that $a_{m',n'} = m' + n'(n'+1)/2$ whenever (m',n') is less than (m,n) in the lexicographic ordering of $\mathbf{N} \times \mathbf{N}$.

- If $n = 0$, by the inductive hypothesis we can conclude

$$a_{m,n} = a_{m-1,n} + 1 = m - 1 + n(n+1)/2 + 1 = m + n(n+1)/2 .$$

- If $n > 0$, by the inductive hypothesis we can conclude

$$a_{m,n} = a_{m-1,n} + 1 = m + n(n-1)/2 + n = m + n(n+1)/2 .$$