CompSci 201, L21: Balanced Binary Search Trees

Logistics, Coming up

- This Wednesday, April 5
  - APT 8 due
- Next Monday, April 10
  - Project P5: Huffman due
- Next Wednesday, April 12
  - APT Quiz 2 due:
  - Covers linked list, sorting, trees
  - No regular APTs this week, just the quiz

Reminder: What is an APT Quiz?

- Set of 3 APT problems, 2 hours to complete.
  - Will be available starting this Saturday afternoon (look for a Sakai/email announcement)
  - Must *complete* by 11:59 pm Wednesday 4/12 (so start before 10)
- Start the quiz on Sakai assessments tool, begins your timer and shows you the link to the problems and submission page.
  - Will look/work just like the regular APT page, just with only 3 problems.

Reminder: What is allowed?

Yes, allowed

- Zybook
- Course notes
- API documentation
- VS Code
- JShell

No, not allowed

- Collaboration or sharing any code.
- Communication about the problems ***at all*** during the window.
- Searching internet, stackoverflow, etc. for solutions.

Reminder: Don’t do these things

1. Do not collaborate. Note that we log all code submissions and will investigate for academic integrity.

2. Do not hard code the test cases (if(input == X) return Y, etc.).

   We show you the test cases to help you debug. But we search for submissions that do this and **you will get a 0 on the APT quiz if you hard code the test cases** instead of solving the problem.

Reminder: How is it graded?

Not curved, adjusted. 3 problems, 10 points each.

Raw score R out of 30.	Adjusted score A out of 30.	100 point grade scale
27 <= R <= 30	A = R	90 – 100
24 <= R <= 26	A = 26	~87
21 <= R <= 23	A = 25	~83
18 <= R <= 20	A = 24	80
15 <= R <= 17	A = 23	~77
12 <= R <= 14	A = 22	~73
9 <= R <= 11	A = 21	70
6 <= R <= 8	A = 20	~67
3 <= R <= 5	A = 19	~63
1 <= R <= 2	A = 18	60

Can still get in the B range even if you can't solve one; don't panic!

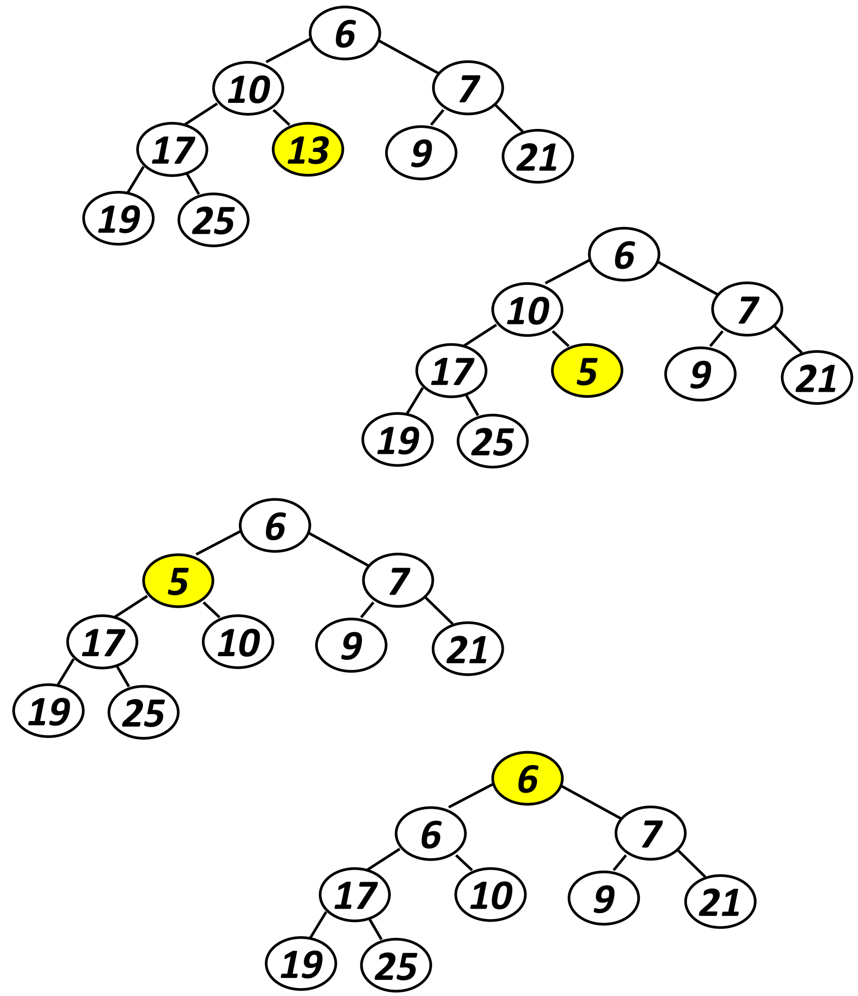
Only going to get a 0 if you collaborate or hard code test cases. Don't do it!

Binary Heap Wrapup

Reminder: You can see a simple DIY implementation of a binary heap-based priority queue at coursework.cs.duke.edu/cs-201-spring-23/diybinaryheap

decreaseKey Operation?

- Suppose we decrease the 13 to 5.
- Violates heap property
- Fix like in the add operation: While violating heap property:
  - Swap with parent



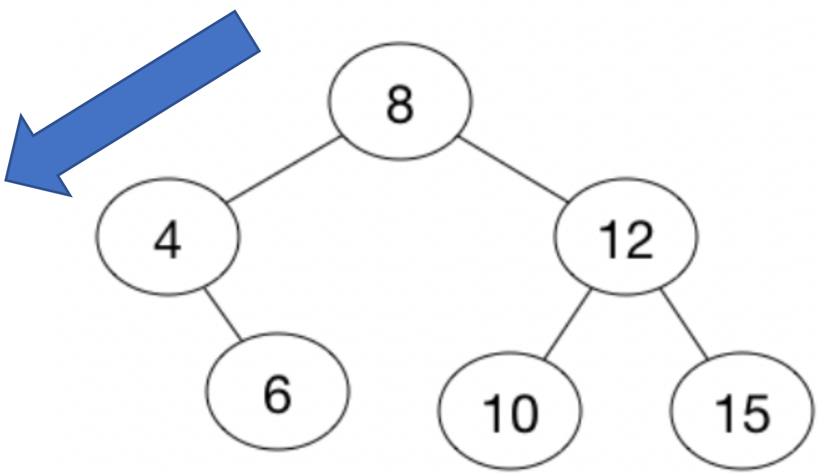
decreaseKey NOT in java.util

- decreaseKey is important for some algorithms, but not supported in many standard libraries (including the java.util PriorityQueue)

- Why not?
  - Note that binary heap does not support efficient *search*
  - In order to do decreaseKey in O(log(n)) time, need to store *references/indices* of all the “nodes.”
  - Adds overhead, not done in java.util

Alternative Implementation: Binary Search Tree

- If your keys happen to be unique…
- Can support O(log(n)) add & remove (smallest) using a binary search tree!
- Smallest is leftmost child



PriorityQueue (with unique keys) using a java.util TreeSet

```java
import java.util.TreeSet;

public class BSTPQ<T extends Comparable<T>> {
    private TreeSet<T> bst;

    public BSTPQ() { bst = new TreeSet<>(); }
    public void add(T element) { bst.add(element); }
    public int size() { return bst.size(); }
    public T peek() { return bst.first(); }

    public T remove() {
        T returnValue = bst.first();
        bst.remove(returnValue);
        return returnValue;
    }

    public void decreaseKey(T oldKey, T newKey) {
        bst.remove(oldKey);
        bst.add(newKey);
    }
}
```

`first` gives smallest element in TreeSet in O(log(n)) time

Can `decreaseKey` by removing and then re-adding, both O(log(n)) time for a TreeSet

Disadvantages to using a Binary Search Tree for your priority queue?

1. All elements must be unique

2. Not array-based, uses more memory and has higher constant factors on runtime

3. Much harder to implement with guarantees that the tree will be balanced.

???

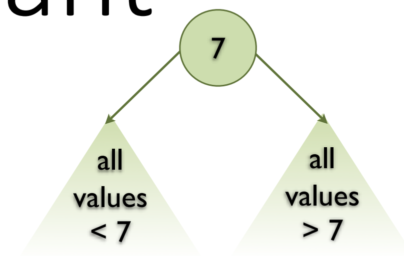
Binary Search Tree Review and Runtime

See videos of live coding a DIYTreeSet as a binary search tree:

- Part 1: Getting started, traversal, iterator
- Part 2: add and contains

And here is the code: coursework.cs.duke.edu/cs-201-spring-23/diytreeset

Binary Search Tree Invariant

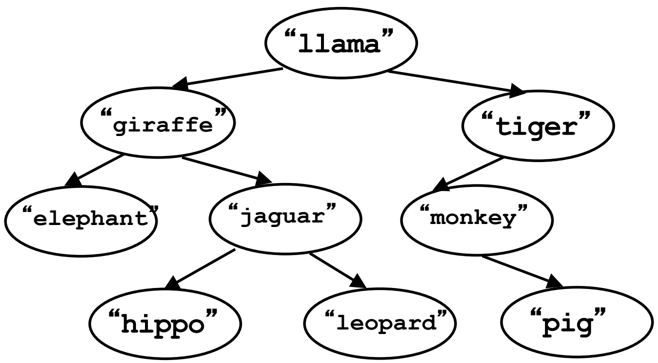


A binary tree is a binary **search** tree if *for every node*:

- Left subtree values are all less than the node's value

AND

- Right subtree values are all greater than the node's value



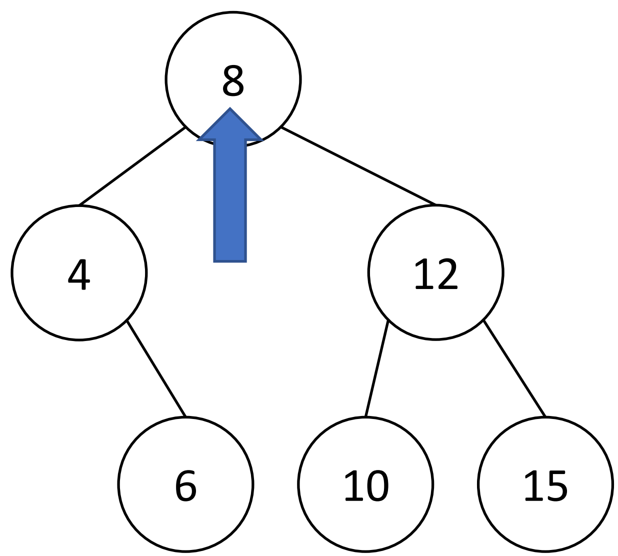
According to some ordering (comparable or comparator)

Enables efficient search, similar to binary search!

Recursive `search`, pictures, pseudocode

```
boolean search(int x, TreeNode t) {
```

- If `t == null`: Return `false`
- If `x == t.info`: Return `true`
- If `x < t.info`: search left
- Else: search right



Searching for 10

10 > 8, so search right

Recursive `search`, pictures, pseudocode

```
boolean search(int x, TreeNode t) {
```

- If `t == null`: Return `false`
- If `x == t.info`: Return `true`
- If `x < t.info`: search left
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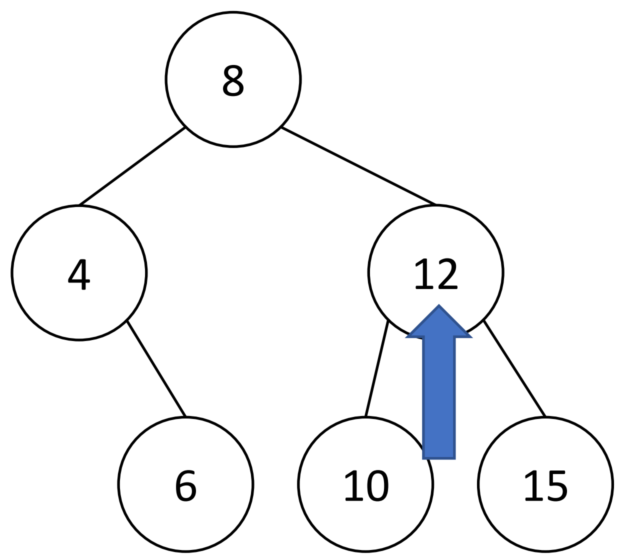
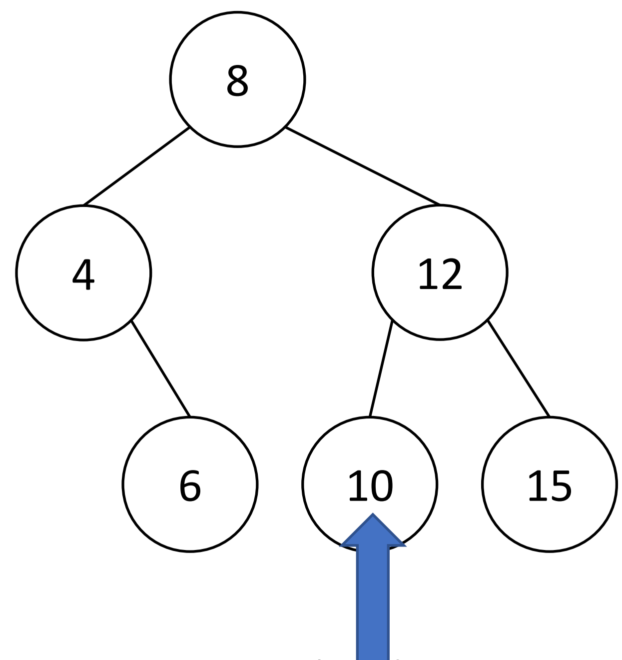


10 < 12 so search left

Recursive `search`, pictures, pseudocode

```
boolean search(int x, TreeNode t) {
```

- If `t == null`: Return `false`
- If `x == t.info`: Return `true`
- If `x < t.info`: search left
- Else: search right



10 == 10 so return true

Recursive `search` code

```
private boolean search(int x, TreeNode t) {
    if (t == null) {
        return false;
    }
    if (t.info == x) {
        return true;
    }
    if (x < t.info) {
        return search(x, t.left);
    }
    return search(x, t.right);
}
```

Base case: no more tree to search, did not find x

Base case: found x

If x is less, search in left subtree

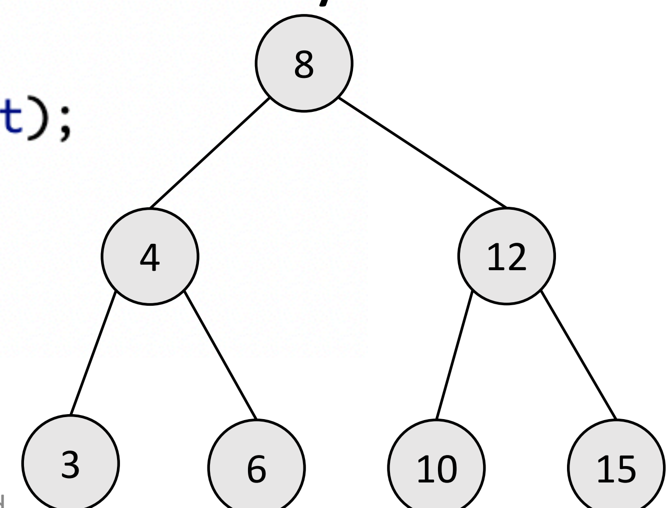
Else search in right subtree

Runtime complexity of BST add/contains on balanced tree

```java
    private boolean search(int x, TreeNode t) {
        if (t == null) {
            return false;
        }
        if (t.info == x) {
            return true;
        }
        if (x < t.info) {
            return search(x, t.left);
        }
        return search(x, t.right);
    }
```

Completely balanced tree:

- $T(N) = T(N/2) + O(1)$
- Solution is **O(log(N))**, same as binary search

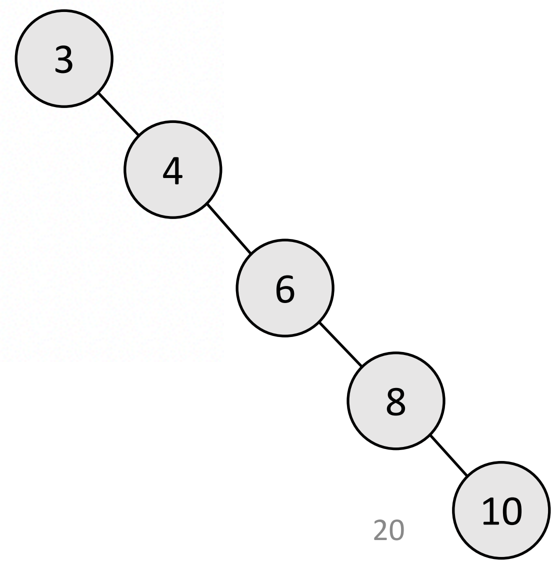


Runtime performance of BST on perfectly **un**balanced tree

```java
private boolean search(int x, TreeNode t) {
    if (t == null) {
        return false;
    }
    if (t.info == x) {
        return true;
    }
    if (x < t.info) {
        return search(x, t.left);
    }
    return search(x, t.right);
}
```

Perfectly unbalanced tree:

- T(N) = T(N-1) + O(1)
- Solution is **O(N)**, search in linked list

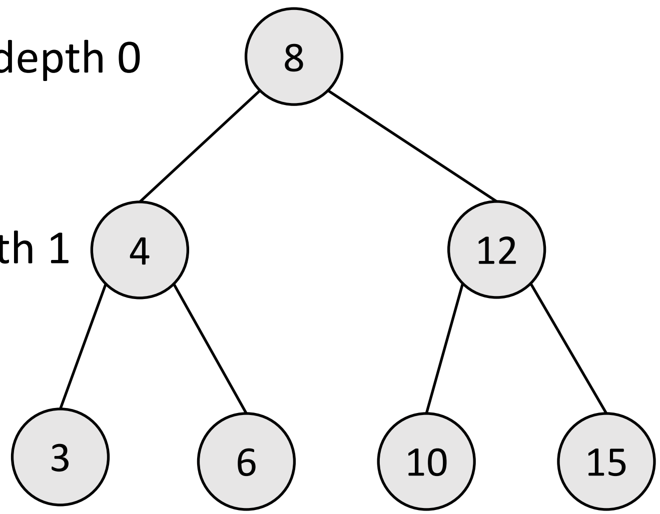


Another perspective: Balanced BST has height $O(\log(n))$

2^0 nodes at depth 0

2^1 nodes at depth 1

2^2 nodes at depth 2



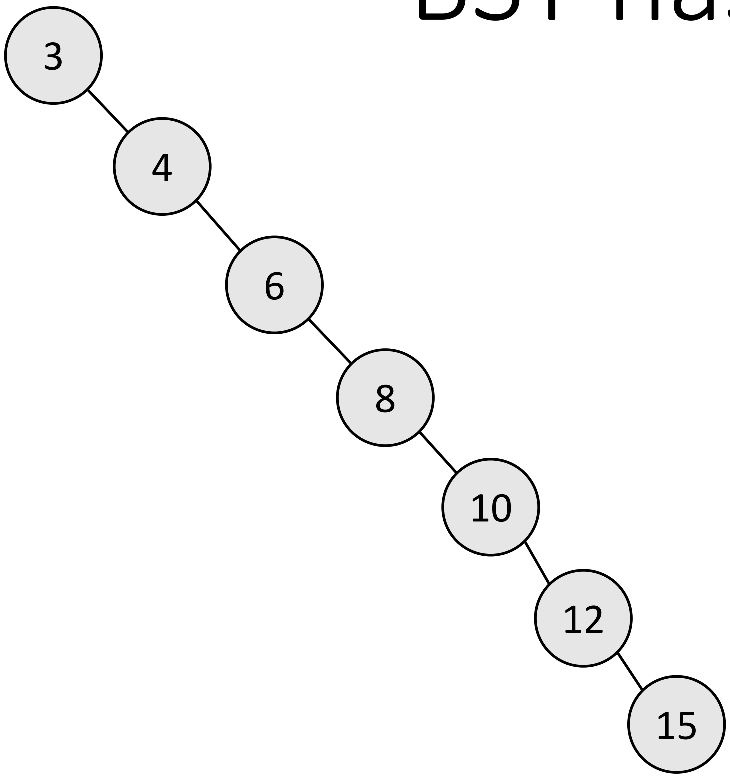
$$n = 2^0 + 2^1 + \cdots + 2^h$$
$$= 2^{h+1} - 1$$
$$\rightarrow h = \log_2(n+1) - 1$$

Geometric series formula:

$$\sum_{i=0}^{k-1} ar^i = \frac{a(1-r^k)}{1-r}$$

Results in $O(\log(n))$ best case performance.

Another perspective: Unbalanced BST has $O(n)$ height



For example, results from:

```
Sort(values)
For each e in values:
  insert(e)
```

Results in $O(n)$ worst case performance.

Experiment: How much difference does it make empirically to do 100,000 random searches?

Timings in milliseconds

See example code in coursework.cs.duke.edu/cs-201-fall-22/diytreeset

N	sorted order DIY binary search tree	random order DIY binary search tree	sorted order java.util TreeSet
1,000	370	4	8
2,000	715	5	11
4,000	1422	5	14
8,000	2905	8	13
16,000	5991	7	12
32,000	Runtime exception	10	13
64,000	…	8	14
…	…	…	…
1,000,000	…	15	24

Average Case: Random Binary Search Tree has $O(\log(n))$ expected height

- Given $x_1, \ldots, x_n$ unique keys
- Let $\sigma(x_1, \ldots, x_n)$ be a uniform random permutation

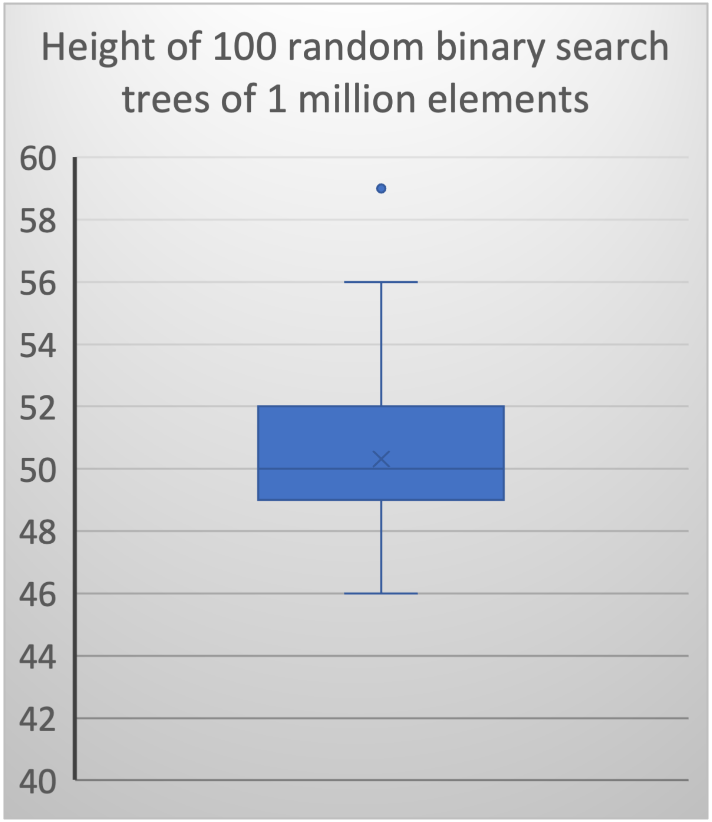
- Theorem 12.4 CLRS (restated):

$\mathbb{E}_\sigma[h_n] \leq \log_2\left(\frac{n^3+6n^2+11n+6}{24}\right)$ is $O(\log(n))$.

Stronger statements about random binary search trees

- At most $h_n \to_{n\to\infty} 4.3 \log_2(n)$ with high probability
  - *Luc Devroye. 1986. A note on the height of binary search trees. J. ACM 33, 3 (July 1986), 489–498. https://doi.org/10.1145/5925.5930*

- Empirical performance. Note that for $n = 1$ million:
  - $2\log_2(n) \approx 40$
  - $3\log_2(n) \approx 60$

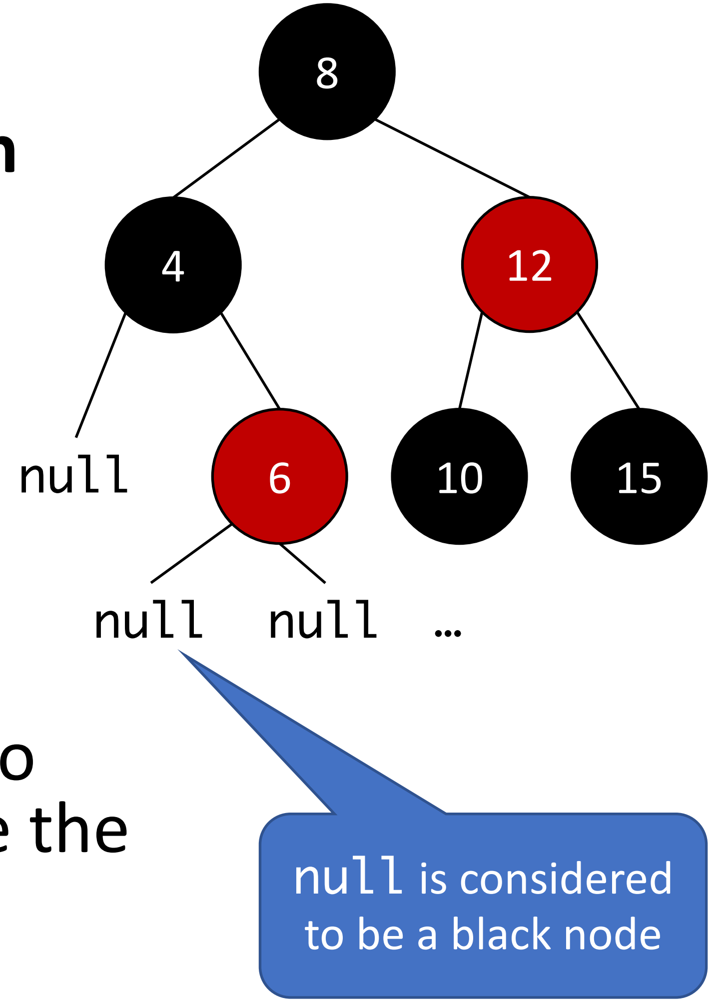


Red-Black Tree: A Balanced Binary Search Tree

Red-Black Tree

Red-Black Trees are **binary search trees** that satisfy the following properties:

1. Every node is red or black,
2. The root is black,
3. A red node cannot have red children, and
4. From a given node, all paths to `null` descendants must have the same number of black nodes.



Red-Black Trees in `java.util`

Class TreeMap<K,V>

java.lang.Object
    java.util.AbstractMap<K,V>
        java.util.TreeMap<K,V>

Type Parameters:

K - the type of keys maintained by this map

V - the type of mapped values

All Implemented Interfaces:

`Serializable`, `Cloneable`, `Map<K,V>`, `NavigableMap<K,`

```
public class TreeMap<K,V>
extends AbstractMap<K,V>
implements NavigableMap<K,V>, Cloneable, Serial
```

A Red-Black tree based `NavigableMap` implementation.

More red-black trees in `java.util`

Class TreeSet<E>

java.lang.Object
    java.util.AbstractCollection<E>
        java.util.AbstractSet<E>
            java.util.TreeSet<E>

Type Parameters:

E - the type of elements maintained by this set

All Implemented Interfaces:

`Serializable`, `Cloneable`, `Iterable<E>`, `Collection<E`

```
public class TreeSet<E>
extends AbstractSet<E>
implements NavigableSet<E>, Cloneable, Serializ
```

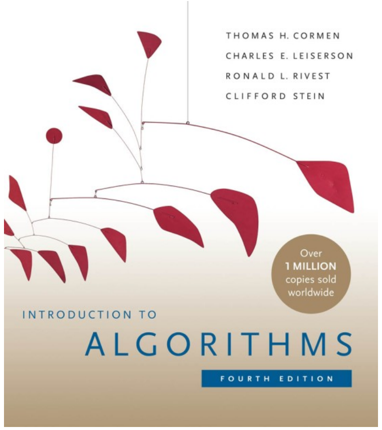
A `NavigableSet` implementation based on a `TreeMap`. T

A “family” tree connection

```
public class TreeMap<K,V>
extends AbstractMap<K,V>
implements NavigableMap<K,V>, Cloneable, Serializable
```

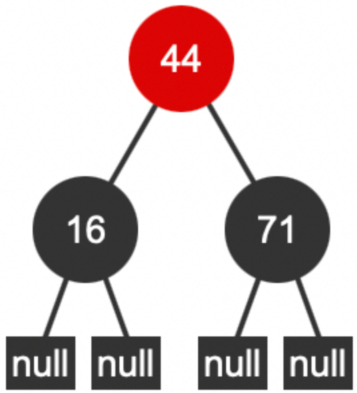
A Red-Black tree based `NavigableMap` implementation. The map is sorted according to the natural ordering of its keys, or by a `Comparator` provided at map creation time, depending on which constructor is used.

This implementation provides guaranteed log(n) time cost for the `containsKey`, `get`, `put` and `remove` operations. Algorithms are adaptations of those in Cormen, Leiserson, and Rivest's *Introduction to Algorithms*.

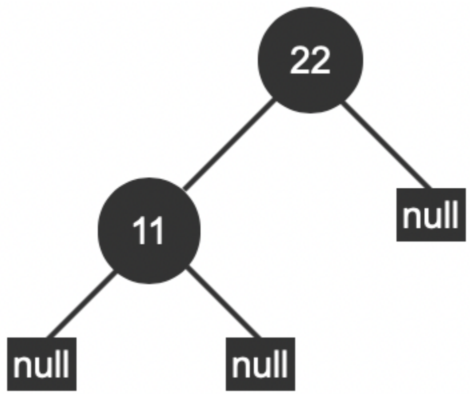


My (doctoral) adviser’s adviser’s adviser (we don’t know each other)

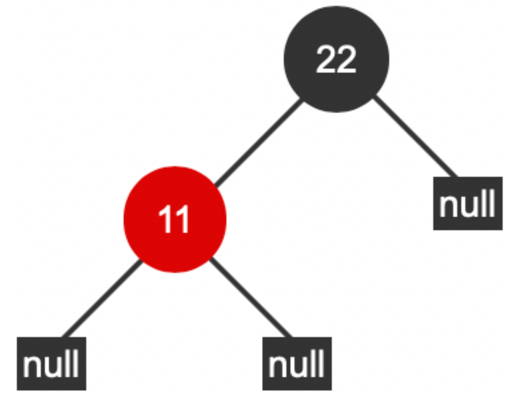
Understanding Red-Black Tree Properties



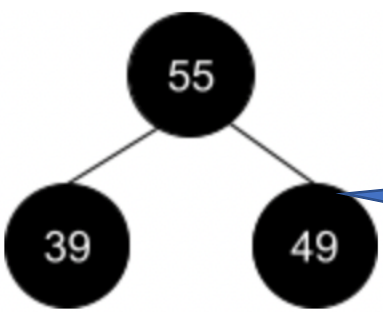
- Root node is red.
- NOT a valid red-black tree.



- Root node is black.
- No red nodes with red children.
- Path (22,null) has 2 black nodes, but path (22,11,null) has 3.
- NOT a valid red-black tree.



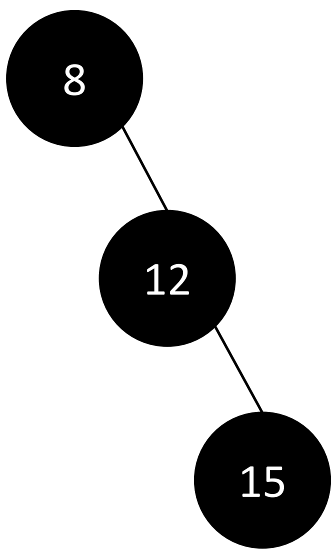
- Root node is black.
- Red node has no red children.
- All paths from a node to null leaves have the same number of black nodes
  - All paths from 22 to null leaves have 2 black nodes.
  - All paths from 11 to null leaves have 1 black node.
- Tree is a valid red-black tree.



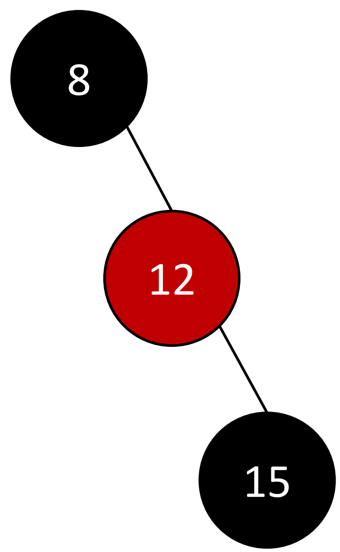
Trick: Not a binary search tree at all!

Reference: ZyBook 21

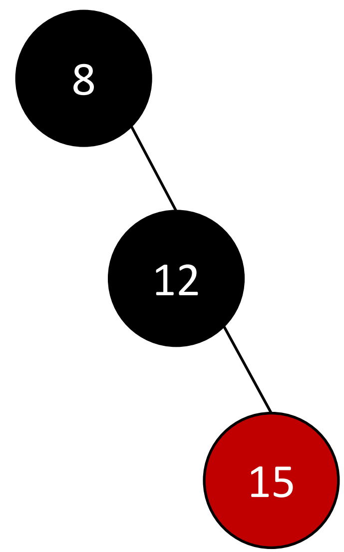
Not all binary search trees **can** be colored as red-black trees



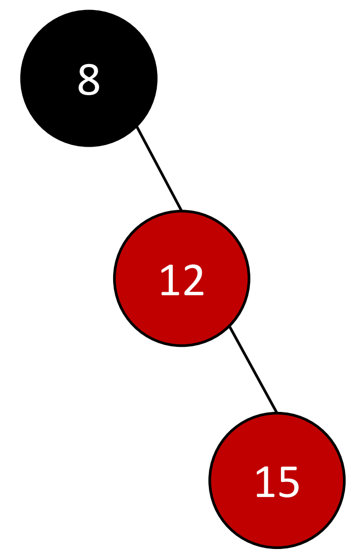
Too many black nodes on right compared to left paths



Too many black nodes on right compared to left paths



Too many black nodes on right compared to left paths



Red node with red child not allowed

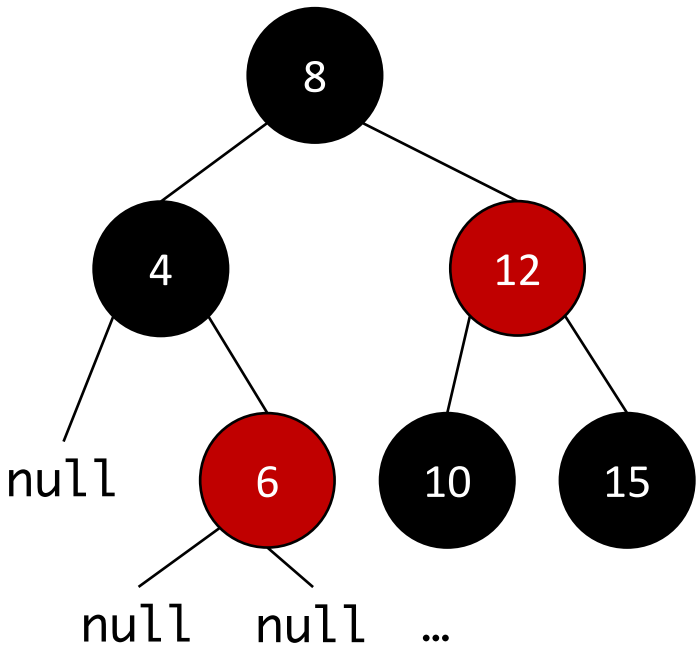
red-black tree properties guarantee approximate balance

- Note that the runtime complexity of add/contains (a.k.a. insert and search) in a binary search tree is proportional to the height of the tree.

- **Claim.** Any red-black tree with N nodes has height that is O(log(N)).

Proof sketch (not going to sweat the details)

1. At least half of the nodes on any root to leaf path are black (because red nodes cannot have red children).
2. All root to leaf paths have the same number of black nodes (property 4)
3. 1+2 imply that all root to leaf paths have within a factor 2 of the same number of nodes.



How do Red-Black Trees Work

Remember, red black trees are also binary search trees (BST).

- `contains/search` – Exact same as BST, no change!
- `add/insert` – Two steps:
    1. Run regular BST add/insert
    2. Color the new node red
    3. Fix the tree to reestablish red-black tree properties

RBTreeNode

```java
public class RBTreeNode {
    int info;
    RBTreeNode parent;
    RBTreeNode left;
    RBTreeNode right;
    boolean red;

    RBTreeNode(int x, boolean red) {⋯
    RBTreeNode(int x, RBTreeNode parent, boolean red) {⋯
    RBTreeNode(int x, RBTreeNode lNode, RBTreeNode rNode,
                RBTreeNode parent, boolean red) {
        info = x;
        left = lNode;
        right = rNode;
        this.parent = parent;
        this.red = red;
    }
```

Just like a regular TreeNode except:

- Store parent reference
- Store color

Search is the same

```java
private boolean search(int x, RBTreeNode t) {
    if (t == null) {
        return false;
    }
    if (t.info == x) {
        return true;
    }
    if (x < t.info) {
        return search(x, t.left);
    }
    return search(x, t.right);
}
```

Insert looks the same…

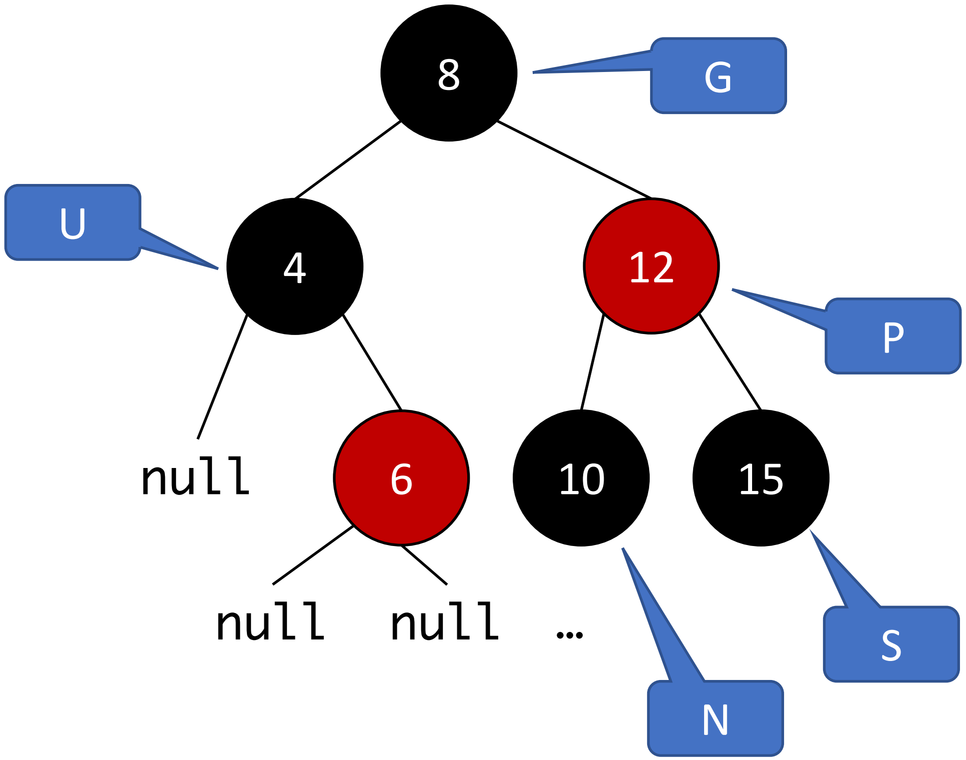
```java
private boolean insert(int x, RBTreeNode t) {
    if (t.info == x) {
        return false;
    }
    if (x < t.info) {
        if (t.left == null) {
            t.left = new RBTreeNode(x, true);
            RBTreeBalance(t.left);
            return true;
        }
        return insert(x, t.left);
    }
    if (t.right == null) {
        t.right = new RBTreeNode(x, true);
        RBTreeBalance(t.right);
        return true;
    }
    return insert(x, t.right);
}
```

Except for the root, always add new nodes as red initially

Need to re-balance (reestablish red-black tree properties) after insertion.

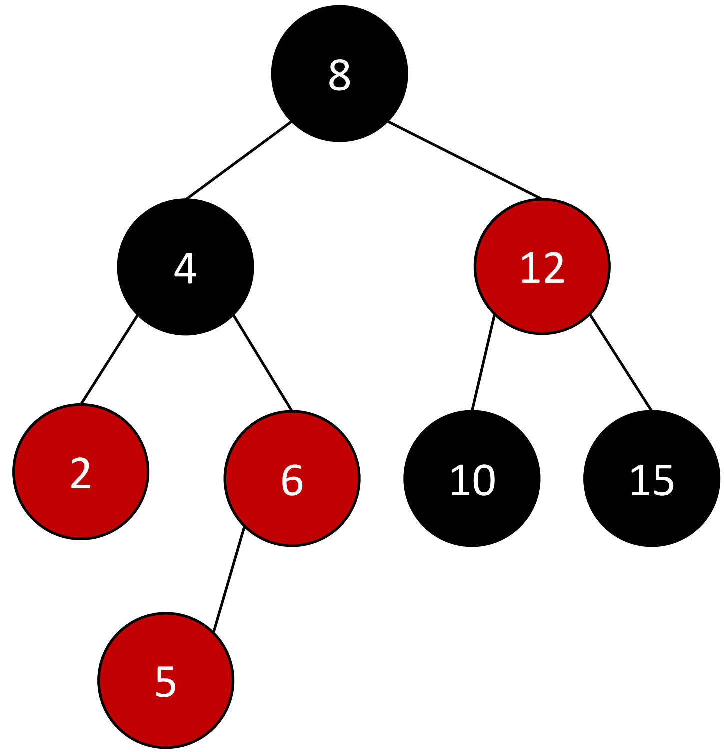
Terminology: Getting to know the family "Tree"

- Node (N)
- Sibling (S)
- Parent (P)
- Grandparent (G)
- Uncle (U)



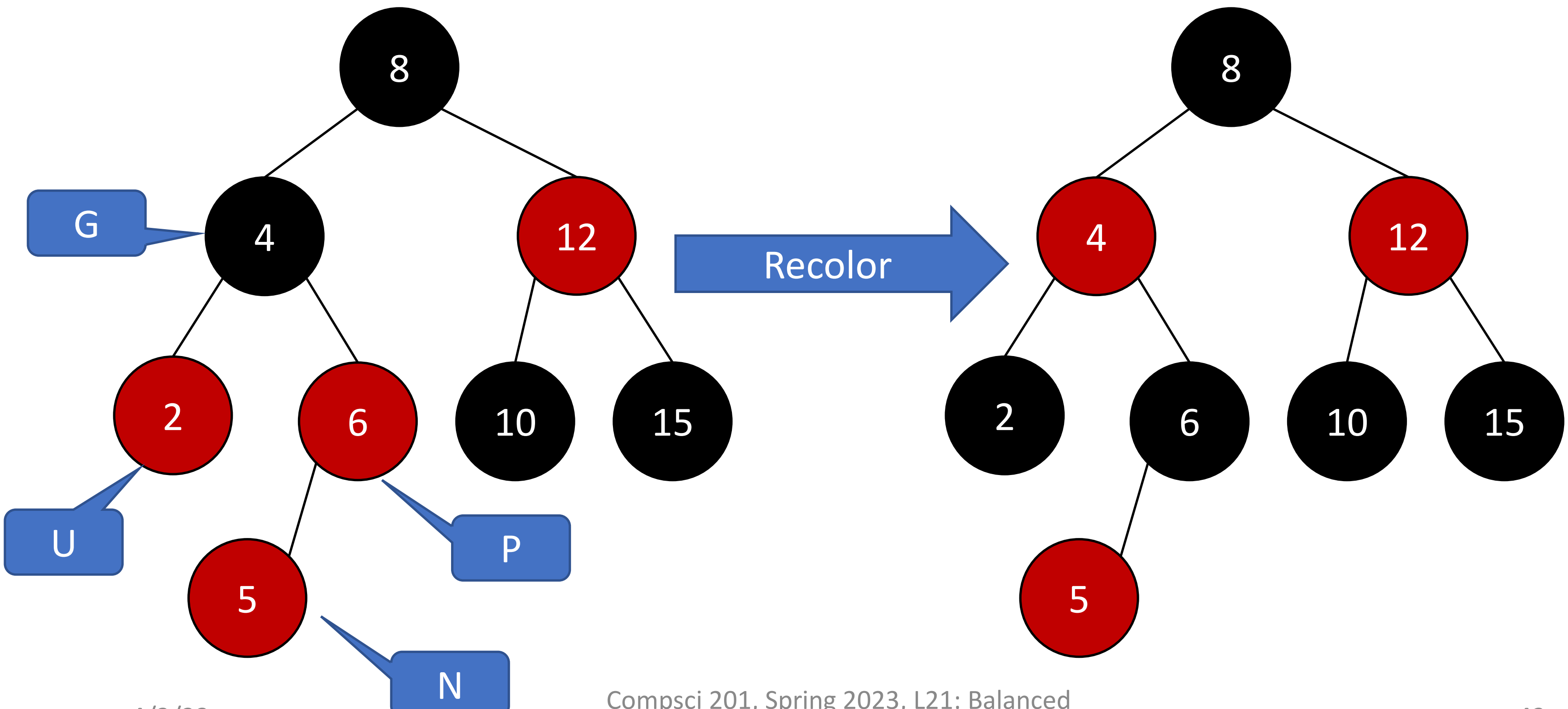
Recoloring

- We always insert a new node as red, see the 5 node here.
  - (This way never violates the black nodes on paths property)
- Violates RBT property: red child of red parent.
- Fix by recoloring?



Recoloring

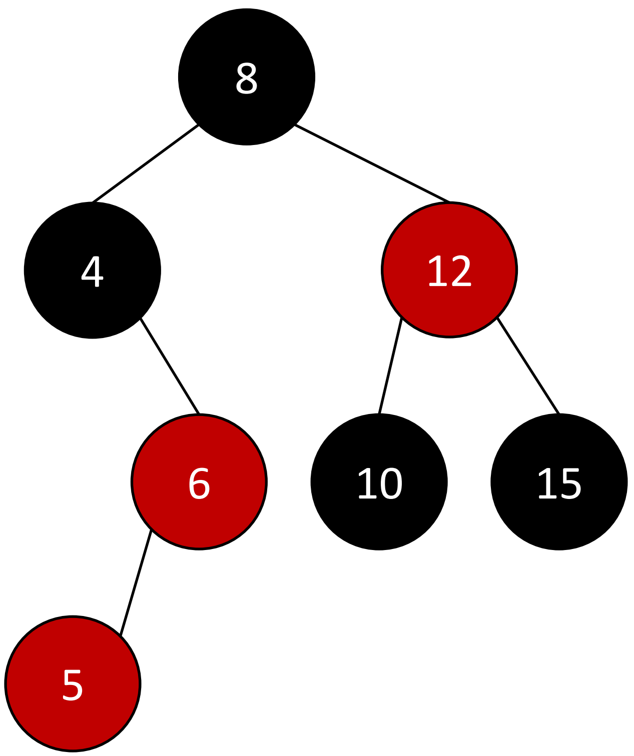
If parent and uncle of new node are both red, color both black and color grandparent red.



Can't fix all problems by recoloring

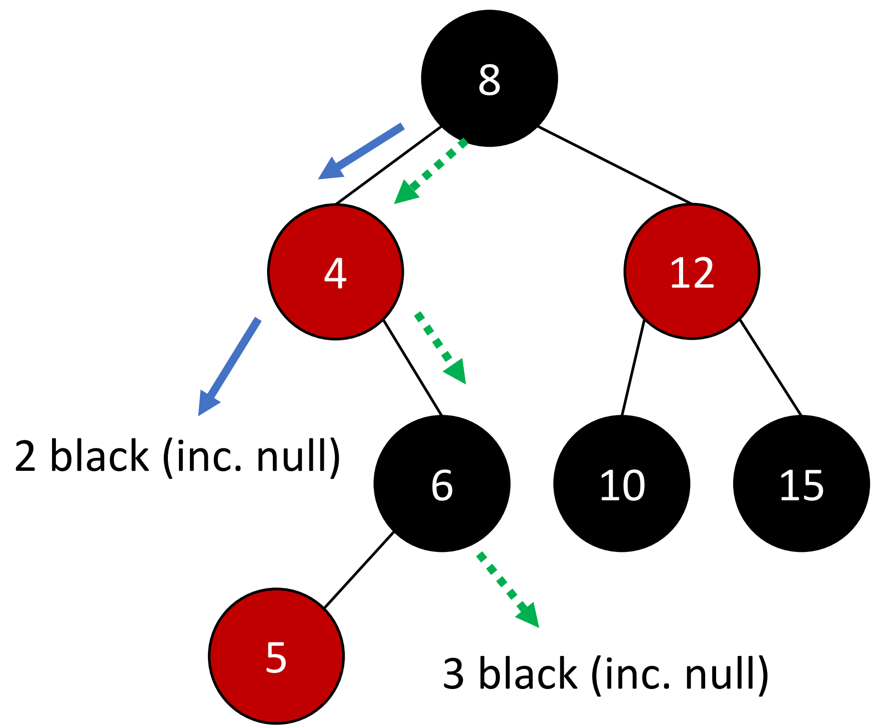
- Suppose we just inserted 5 here.

- "Looks" like we could just recolor…
  - Set 4 red, 6 black?

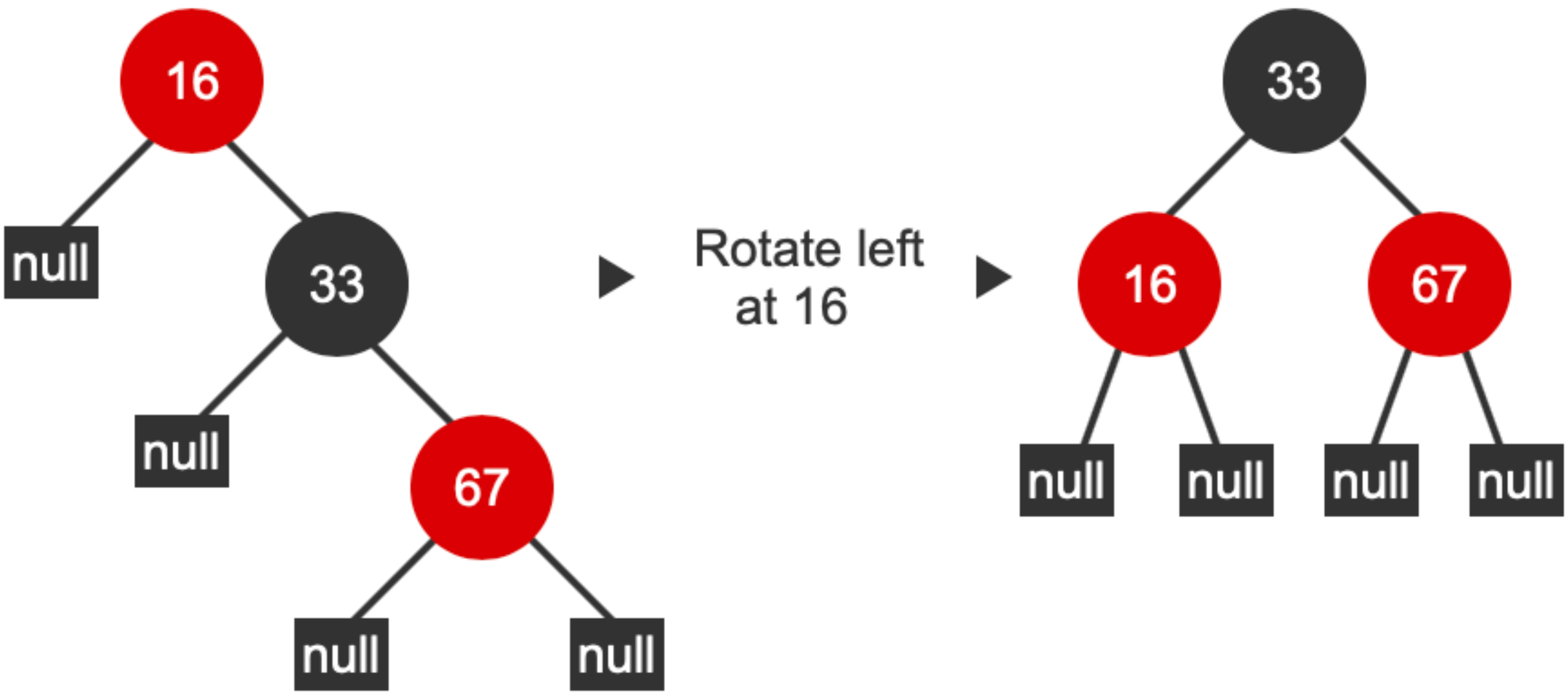


Sometimes need to rotate

- Looks good, but…
- Violating path property now. Need to **rotate**: actually change the tree structure.



Left Rotation

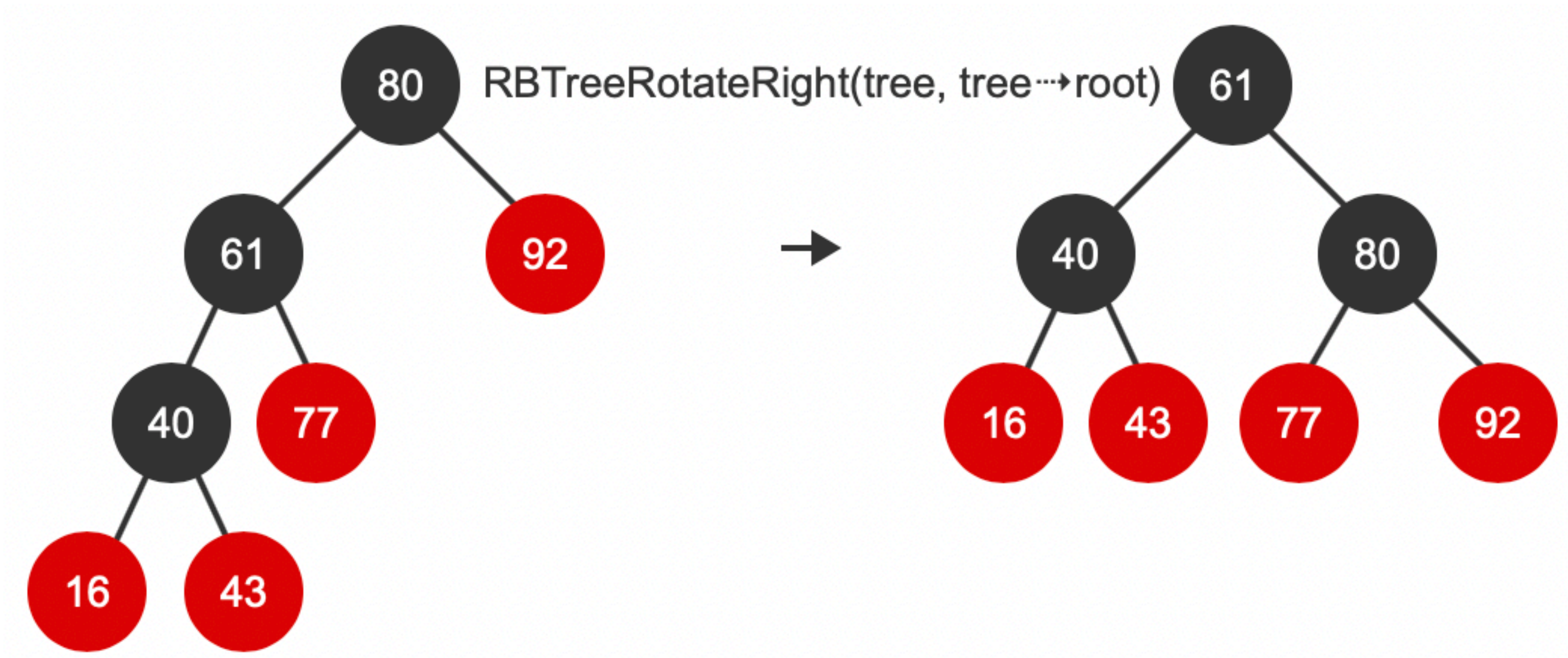


Not a valid red-black tree

Valid red-black tree

ZyBook 21

Right Rotation



ZyBook 21

Case Analysis

- Full rebalance algorithm proceeds by cases:
  - Cases vary by color and position of node, parent, grandparent, uncle.
  - Deal with cases by recoloring, left rotations, and right rotations.
- Remove has case analysis as well.

- Want the details? See the ZyBook (or CLRS Intro to Algorithms for the standard reference).