Due on March 30, 2020
60 points total

General directions:

All answers to non-programming questions must be typed, preferably using LaTeX. If you are unfamiliar with LaTeX, you are strongly encouraged to learn it. However, answers typed in other text processing software and properly converted to a PDF file will also be accepted. To submit your file, upload your PDF on Gradescope for assignment “HW X (PDF).” Handwritten answers or PDF files that cannot be opened will not be graded and will not receive any credit.

Finally, please read the detailed collaboration policy given on the course website. You are **not** allowed to discuss homework problems in groups of more than 3 students. **Failure to adhere to these guidelines will be promptly reported to the relevant authority without exception.**

Point values: Every problem has a specified amount of points which are awarded for the correctness of your solutions. In addition, each proof-oriented problem has an additional **style point**. In the homework handout, this is signified by a ”+1” in the point value. To earn this point, your solutions should be clear, well organized, and easy to follow. This is to encourage not only perfectly correct solutions, but well presented ones.

Problem 1 (24+6 points)
For each of the relations below, defined on $\mathbb{R}^+ \times \mathbb{R}^+$ where $\mathbb{R}^+$ is the set of positive real numbers, prove or disprove that (i) the relation is a partial order and that (ii) the relation is a total order:

(a) (4 + 4 + 2 points) The relation is given by the set:

$$\{((a,b),(c,d)) : \sqrt{a^2+b^2} < \sqrt{c^2+d^2} \text{ or } a < c\}.$$

(b) (4 + 4 + 2 points) The relation is given by the set:

$$\{((a,b),(c,d)) : \sqrt{a^2+b^2} < \sqrt{c^2+d^2} \text{ or } \left(\sqrt{a^2+b^2} = \sqrt{c^2+d^2} \text{ and } a < c\right)\}.$$

(c) (4 + 4 + 2 points) The relation is given by the set:

$$\{((a,b),(c,d)) : \sqrt{a^2+b^2} < \sqrt{c^2+d^2} \text{ or } \left(\sqrt{a^2+b^2} = \sqrt{c^2+d^2} \text{ and } \tan^{-1}(b/a) < \tan^{-1}(d/c)\right)\}.$$

Problem 2 (13+2 points)
Let $T = (V, E)$ be a tree with n vertices and at least one edge. For any positive integer k, let y_k denote the number of vertices with degree k.

(a) (9+1 points) Prove that the following equality holds:

$$y_1 = 2 + \sum_{k=3}^{n-1} (k-2) \cdot y_k.$$

(b) (4+1 points) Use part (a) to prove the following: if T is a rooted full binary tree, then T has $(n+1)/2$ leaves. (Recall that a *leaf* of a rooted tree is a node with zero children. Also, in a rooted full binary tree, every node has zero or two children.)

Problem 3 (14+1 points)
A graph is said to be k-colorable if it is possible to assign each vertex to one of k colors such that the two endpoints of every edge are assigned different colors. Prove that a graph $G = (V, E)$ is 2^k-colorable if and only if E can be partitioned into k sets $E_1, \ldots, E_k$ such that for every $1 \leq i \leq k$, $G_i = (V, E_i)$ is a bipartite graph.