Contents

Chapter 1

Sets and Sequences

1.1 Sets

1.1.1 Overview

This lecture begins our second module: Sets, Maps, and Sequences. (The first module was Proofs and Mathematical Logic.) In this lecture, we give an overview of sets, operators on sets, rules that we can apply, and the set builder notation.

1.1.2 Sets

Giving a precise definition of a *set* is a notoriously difficult problem (Russel's paradox). However, for the purposes of this class, we can simply think of a set as a collection of objects known as "elements" of the set. Note that a set never has repeated elements, and elements of a set are not ordered. For example,

$$S = \{1, 4, 3\}$$

is a set containing 3 elements. Equivalent versions of S include $\{1, 1, 4, 3\}$ and $\{4, 3, 1\}$. One thing to note is that $A = \{\emptyset\}$ is not an empty set - it is a set containing the empty set.

Definition 1. *If A and B are sets, then A is a* subset *of B (denoted by $A \subseteq B$) if the following proposition is true:*

$$x \in A \Rightarrow x \in B.$$

This definition is fairly intuitive: a subset of a set is a new set whose elements all belong to the original set. For example, $\{1\}$ is a subset of S (defined above), but $\{1, 2\}$ is not. Also, note that "1" is not a subset of S, because "1" on its own is not a set.

If A is the empty set (which we denote by $A = \emptyset$), that is, A contains no elements, then A is a subset of any set. This is because the implication above that defines "subset" is vacuously true: $x \in A$ is false for any x, and FALSE$\Rightarrow Q$ is TRUE regardless of the value of Q.

We can derive from the definition of a subset that for sets A and B, $A = B$ iff $A \subseteq B$ and $B \subseteq A$.

Definition 2. *If A and B are sets, then A is a* proper subset *of B (denoted $A \subset B$) if A is a subset of B and $A \neq B$.*

In other words, A is a proper subset of B if every element of A is in B, and B contains at least one element that is not in A. Note that the empty set is a proper subset of any non-empty set, and in the above example, $\{1\}$ is a proper subset of S.

Definition 3. *The* power set *of a set S is a set of all the subsets of S, often denoted by 2^S.*

As discussed above, the empty set is a subset of any set. Furthermore, any set is a subset of itself. Thus, the power set of S is the following:

$$2^S = \{\emptyset, \{1\}, \{4\}, \{3\}, \{1,4\}, \{1,3\}, \{4,3\}, S\}.$$

The set 2^S has 8 elements, each of which is a subset of S.

Definition 4. *The* cardinality *of a finite set is the number of elements in it, often denoted by $|S|$.*

(A finite set is simply a set that contains a finite number of elements.) Continuing with our example, we have $|S| = 3$ and $\left|2^S\right| = 8$. It is unclear how we can define cardinality of an infinite set, but as we will see how we can still impose a notion of size on infinite sets in a later lecture.

Definition 5. *Sets A and B are said to be disjoint if $A \cap B = \emptyset$.*

For example, set $A = \{1,2\}$ and $B = \{3,4\}$ are disjoint.

1.1.3 The Set Builder Notation

One way to describe a set is to explicitly write out its elements in curly brackets, such as $S = \{1,4,3\}$ from above. However, this method is cumbersome when $|S|$ is large, and completely fails when S is infinite. To combat this, and for the sake of clarity, we often use the set builder notation, which we now describe. Consider the following set:

$$A = \{x \in \mathbb{Z}^+ : \exists y \in \mathbb{Z}^+ . \, x = 2y\}.$$

(Note: we sometimes use a "$|$" instead of a ":".) Here, A is the set of positive even integers. Notice that we are not explicitly stating which numbers belong to A, but rather, we give a predicate formula that determines whether a number belongs to A. Another example is the following, which defines a circle of radius r:

$$C = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = r^2\}.$$

You can probably guess what $\mathbb{R}^2$ represents: it is the set of all points (x,y), where x and y are real numbers. Note that unlike in sets, order matters here: the point $(3,5)$ is not equal to $(5,3)$.

1.1.4 Operators on Sets

In propositional logic, we had operators such as $\wedge$ and $\vee$ that acted on propositional variables. Similarly, we will use three basic operators on sets, and the definition of each agrees with our intuitive understanding of the corresponding words. We shall keep the following example in mind as a running example, where the universe is the set of positive integers less than 7:

$$A = \{1,3,4\}, \quad B = \{2,6\}.$$

1. Union: The *union* of two sets A and B, denoted $A \cup B$, is a set defined as follows:

$$x \in A \cup B \Leftrightarrow (x \in A \vee x \in B).$$

   In our example, $A \cup B = \{1, 2, 3, 4, 6\}$.

2. Intersection: The *intersection* of two sets A and B, denoted $A \cap B$, is a set defined as follows:

$$x \in A \cap B \Leftrightarrow (x \in A \wedge x \in B).$$

   In our example, $A \cap B = \emptyset$.

3. Complement: The *complement* of a set A, denoted by $\overline{A}$ or A^c, is defined as follows:

$$x \in \overline{A} \Leftrightarrow x \notin A.$$

   (The notation $x \notin A$ is shorthand for $\neg(x \in A)$; in other words, it means x is not an element of A.) In our example, $\overline{A} = \{2, 5, 6\}$ and $\overline{B} = \{1, 3, 4, 5\}$.

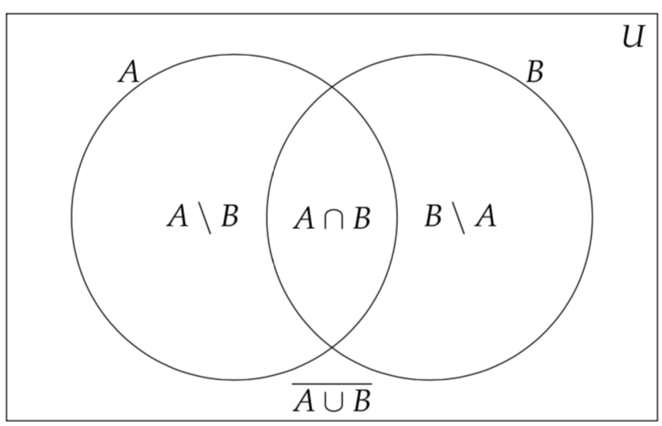
Now we introduce a fourth operator: the *difference* of sets A and B, denoted by $A \setminus B$ or $A - B$, is defined as follows:

$$x \in A \setminus B \Leftrightarrow (x \in A \wedge x \notin B).$$

Why did we isolate this operator from the other three? Because this operator is not actually new—it can be derived from previous operators: $A \setminus B = A \cap \overline{B}$. We can intuitively justify this equivalence, but we will develop the ideas necessary to formally prove it a little later.

1.1.5 Venn Diagrams

Although Venn diagrams do *not* give formal proofs of set equivalences, they are often helpful when visualizing set operations.



If A and B and sets, then the circle on the left represents A and the circle on the right represents B. This diagram contains 4 disjoint regions, representing elements exclusively in A, exclusively in B, in both, or in neither. By using a Venn diagram, we can visualize many equivalences for sets, including the rules that we describe in the following section. We can also easily conjecture some equivalences for sets such as

$$B = (A \cap B) \cup (B \setminus A) \quad \text{and} \quad A \cap B = (A \cup B \setminus (A \setminus B)) \setminus (B \setminus A).$$

A similar diagram can be drawn for 3 sets, and such a diagram would have 8 disjoint regions. Notice that this has a combinatorial interpretation: for each set, an element can either be in or out of that set. Thus, there are $2^3 = 8$ disjoint regions.

1.1.6 Rules for Sets

Now we consider some rules that we can apply on sets, all of which should be familiar:

- Commutativity: $A \cup B = B \cup A$ and $A \cap B = B \cap A$.
- Associativity: $(A \cup B) \cup C = A \cup (B \cup C)$ and $(A \cap B) \cap C = A \cap (B \cap C)$.
- Distributivity: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and $A \cup (B \cap C) = (A \cup B) \cap (A \cap C)$.
- Idempotence: $A \cap A = A$ and $A \cup A = A$.
- Complement: $\overline{\overline{A}} = A$.

We've already seen two versions of De Morgan's laws: one for propositional logic, and one for predicate logic. Now we'll see the version of De Morgan's Laws for sets:

- The complement of the intersection is the union of the complements, i.e., $\overline{A \cap B} = \overline{A} \cup \overline{B}$.
- The complement of the union is the intersection of the complements, i.e., $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

To prove these laws, we will (again) use De Morgan's laws for propositional logic. In general, to show that two sets X and Y are equal, we must prove that every element of X belongs to Y and vice versa. So we start by assuming $x \in \overline{A \cap B}$ and apply rules:

$$\begin{aligned}
& x \in \overline{A \cap B} & \\
& \Leftrightarrow x \notin A \cap B & \text{(definition of complement)} \\
& \Leftrightarrow \neg(x \in A \cap B) & \text{(definition of } \notin) \\
& \Leftrightarrow \neg(x \in A \wedge x \in B) & \text{(definition of } \cap) \\
& \Leftrightarrow \neg(x \in A) \vee \neg(x \in B) & \text{(De Morgan's for } \wedge) \\
& \Leftrightarrow (x \in \overline{A}) \vee (x \in \overline{B}) & \text{(definition of complement)} \\
& \Leftrightarrow x \in \overline{A} \cup \overline{B}. & \text{(definition of } \cup)
\end{aligned}$$

Since every line above is equivalent to the previous line, we have simultaneously shown both $\overline{A \cap B} \subseteq (\overline{A} \cup \overline{B})$ and $(\overline{A} \cup \overline{B}) \subseteq \overline{A \cap B}$, as desired. Now we do the same thing for the other De

Morgan's law, by assuming $x \in \overline{A \cup B}$ and applying rules:

$$\begin{aligned}
& x \in \overline{A \cup B} & \\
& \Leftrightarrow x \notin A \cup B & \text{(definition of complement)} \\
& \Leftrightarrow \neg(x \in A \cup B) & \text{(definition of } \notin) \\
& \Leftrightarrow \neg(x \in A \vee x \in B) & \text{(definition of } \cup) \\
& \Leftrightarrow \neg(x \in A) \wedge \neg(x \in B) & \text{(De Morgan's for } \vee) \\
& \Leftrightarrow (x \in \overline{A}) \wedge (x \in \overline{B}) & \text{(definition of complement)} \\
& \Leftrightarrow x \in \overline{A} \cap \overline{B}. & \text{(definition of } \cap)
\end{aligned}$$

Notice that if we think of $\cup$ as $\vee$ and $\cap$ as $\wedge$ (the symbols even resemble each other), then these rules are exactly the same as the rules for propositional logic.

1.1.7 Cartesian Products

Definition 6. *The* Cartesian product *of sets A and B, denoted by $A \times B$, is a set defined as follows:*

$$A \times B = \{(a, b) : a \in A, b \in B\}.$$

Notice that each element of $A \times B$ is a sequence containing two elements—the first from A, and the second from B. For example,

$$A = \{1, 5, 3\}, \quad B = \{0, 2\}.$$

Then

$$A \times B = \{(1, 0), (1, 2), (5, 0), (5, 2), (3, 0), (3, 2)\}.$$

Remember than in any set, we can freely alter the order of elements, but in a sequence, we cannot alter the order of elements. Thus, we can also write $A \times B$ as

$$A \times B = \{(5, 0), (1, 2), (3, 2), (5, 2), (3, 0), (1, 0)\},$$

but since $(0, 1)$ is not an element of $A \times B$, we have

$$A \times B \neq \{(0, 1), (1, 2), (5, 0), (5, 2), (3, 0), (3, 2)\}.$$

Observe that $\mathbb{R}^2$ is shorthand for $\mathbb{R} \times \mathbb{R}$, and it represents the Cartesian plane that we often use to plot functions such as $y = 3x^2 + 1$.

Suppose A and B are finite sets. Recall that the cardinality of set A, denoted $|A|$, is the number of elements in A; similarly, $|B|$ is the cardinality of set B. What is the cardinality of their Cartesian product, i.e., $|A \times B|$? There are $|A|$ choices for the first term of each pair and $|B|$ choices for the second term of each pair. Thus, for finite sets A, B:

$$|A \times B| = |A| \cdot |B|.$$

We can also take the Cartesian product of a set and itself. A common notation used for expressing this is as follows:

$$A^n = \underbrace{A \times A \times ..... \times A}_{\text{n times}}.$$

1.1.8 Common Sets

The following special symbols are used to denote common sets:

$\mathbb{Z}$: the set of integers
$\mathbb{Q}$: the set of rational numbers
$\mathbb{R}$: the set of real numbers
$\mathbb{C}$: the set of complex numbers

Notice that we have the hierarchy $\mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$. A superscript $^+$ added to any of the above symbols (except $\mathbb{C}$), restricts the set to its positive elements. For example, $\mathbb{Z}^+$ denotes the set of positive integers. Similarly, $\mathbb{Z}^-$ denotes the set of negative integers. If we want to refer to the set of non-negative integers, we can use $\mathbb{Z}_0^+$. Finally, the set of positive integers $\mathbb{Z}^+$ is also known as the natural numbers, and this set is denoted by the symbol $\mathbb{N}$.

1.1.9 Summary

In this lecture, we introduced the notion of sets and basic operators on sets. We also saw Venn diagrams, rules (such as De Morgan's) for these operators, and the set builder notation. Finally, we took a look at notation used to refer to common sets.

Chapter 2

Relations

2.1 Relations

Given two sets A and B, we often want to describe the ways in which the elements of A and B relate to each other. For example, if A denotes the set of students at a college, and B denotes the set of classes, then a student in A could relate to a class in B if the student is taking that class.

Definition 7. *A* binary relation *between two sets A and B is a set R of ordered pairs (a, b) consisting of elements $a \in A$ and $b \in B$. In other words, $R \subseteq A \times B$. If $(a, b) \in R$, we often write it as aRb.*

Example 1: Let $A = B = \mathbb{Z}^+$ and define the following relation R: a relates to b if and only if $b = 2a$. Thus, $R \subseteq \mathbb{Z}^+ \times \mathbb{Z}^+$. We can explicitly write down elements of R:

$$R = \{(1, 2), (2, 4), (3, 6), \dots\}.$$

In set builder notation, this is

$$R = \{(a, 2a) : a \in \mathbb{Z}^+\}.$$

Note that for every relation, we may not be able to describe it with a similar succinct formulation. Any subset of the Cartesian product is a valid binary relation.

Example 2: Suppose $A = \{1, 2, 3, 4\}$, $B = \{a, b, c, d, e, f\}$, and relation $R = \{(1, a), (2, c), (3, f), (3, c)\}$. We can draw this relation as a map from A to B, given in Figure 2.1.

One common relation that is prevalent throughout mathematics is known as a function.

Definition 8. *A* function *$f : A \to B$ is a relation on A and B that satisfies the following: for every $a \in A$, there exists exactly one $b \in B$ such that (a, b) relates to b under f. In this case, we write $f(a) = b$.*

Definition 9. *Let R be a relation on sets A and B. The set A is known as the* domain *of R, B is the* co-domain *of R, and the* range *of R is the set $\{b \in B | \exists a \in A.(a, b) \in R\}$.*

Definition 10. *Let f be a function from A to B.*

1. *f is* injective *("into") if for every $b \in B$, there exists at most one $a \in A$ such that $f(a) = b$.*
2. *f is* surjective *("onto") if its range equals B.*
3. *f is* bijective *("one-to-one") if it is injective and surjective.*

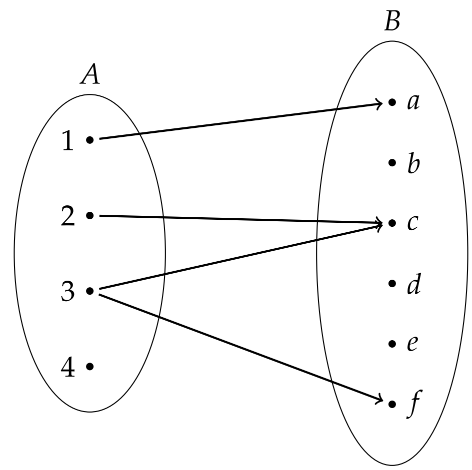


Figure 2.1: Figure representing relation in Example 2.

property of relation	type of relation
$\forall a \in A, \lvert\{b \in B \lvert (a,b) \in R\}\rvert = 1$:	function
$\forall b \in B, \lvert\{a \in A \lvert (a,b) \in R\}\rvert \leq 1$:	"INTO" (injective)
$\forall b \in B, \lvert\{a \in A \lvert (a,b) \in R\}\rvert \geq 1$:	"ONTO" (surjective)

Table 2.1: Classes of relations

2.1.1 Relations on Sets

Now we consider the case where the domain and co-domain are the same set.

Definition 11. *A relation $R \subseteq A^2$ is* reflexive *if, for every $a \in A$, $(a,a) \in R$.*

Example:

1. $R \subseteq \mathbb{R}^2$. $R = \{(x,y) \in \mathbb{R}^2 : x = y\}$ is reflexive.
2. $R \subseteq \mathbb{R}^2$. $R = \{(x,y) \in \mathbb{R}^2 : x \geq y\}$ is reflexive.
3. $R \subseteq \mathbb{R}^2$. $R = \{(x,y) \in \mathbb{R}^2 : x = y, \forall x \in Q\}$ is not reflexive.

Definition 12. *A relation $R \subseteq A^2$ is* symmetric *if, for every $a,b \in A$, $(a,b) \in R$ implies $(b,a) \in R$.*

Example:

1. $R \subseteq \mathbb{R}^2$. $R = \{(x,y) \in \mathbb{R}^2 : 0 \leq x,y \leq 1\}$ is symmetric.
2. $R \subseteq \mathbb{R}^2$. $R = \{(x,y) \in \mathbb{R}^2 : x = y\}$ is symmetric.
3. $R = \emptyset$ is symmetric.

Definition 13. *A relation $R \subseteq A^2$ is* transitive *if, for every $a,b,c \in A$, $(a,b) \in \mathbb{R} \wedge (b,c) \in R$ implies $(a,c) \in R$.*

Example:

1. $R \subseteq \mathbb{R}^2$. $R = \{(x,y) \in \mathbb{R}^2 : 0 \leq x, y \leq 1\}$ is transitive.

2. $R \subseteq \mathbb{R}^2$. $R = \{(x,y) \in \mathbb{R}^2 : x \geq y\}$ is transitive.

2.1.2 Equivalence Relations

Definition 14. *A relation $R \subseteq A^2$ is an* equivalence relation *if R is reflexive, symmetric, and transitive.*

Example:

1. $R \subseteq \mathbb{R}^2$. $R = \{(x,y) \in \mathbb{R}^2 : x = y\}$ is an equivalence relation.

   - For all $x \in \mathbb{R}$, $(x,x) \in R$. Therefore it is reflexive.
   - If $(x,y) \in R$, then $x = y$, and thus $(y,x) \in R$. Therefore it is symmetric.
   - If $(x,y), (y,z) \in R$, then $x = y = z$, and thus $(x,z) \in R$. Therefore it is transitive.

2. $\{(x,y) \in \mathbb{Z}^2 : |x-y| \text{ is divisible by } 13\}$ is an equivalence relation.

   - For all $x \in \mathbb{Z}$, $(x,x) \in R$ because $|x-x| = 0$, which is divisible by 13. Therefore it is reflexive.
   - If $(x,y) \in Z$, then $|y-x| = |x-y|$ which is a multiple of 13, and thus $(y,x) \in R$. Therefore it is symmetric.
   - If $(x,y), (y,z) \in Z$, then $|x-y|$ and $|y-z|$ are multiples of 13 and so are $x-y$ and $y-z$. Say $x-y = 13a$ and $y-z = 13b$. Then $x-z = 13(a+b)$ and thus $|x-z|$ is a multiple of 13. Therefore it is transitive.

Partition

One of the reasons that equivalence relations are interesting is because of their connection to partitions. To motivate this connection, let's consider the set $A = \{1,2,3\}$ and define a relation R on 2^A as follows: $(X,Y) \in R$ if and only if $|X| = |Y|$. We shall explicitly draw this relation below: for every element of 2^A, and if X relates to Y, we'll draw an arrow from X to Y.

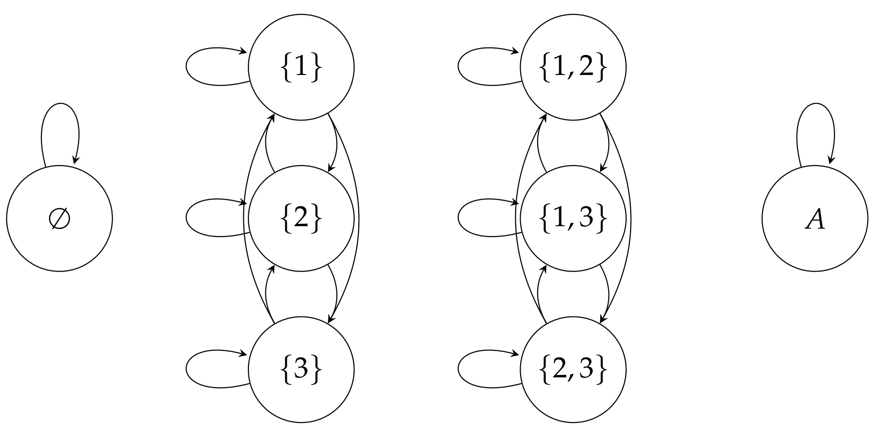


Figure 2.2: The equivalence relation $R = \{(X,Y) : |X| = |Y|\}$ on $A = \{1,2,3\}$, where an arrow from S_1 to S_2 indicates $(S_1, S_2) \in R$.

In Fig. 2.2, each circle represents an element of 2^A (i.e., a subset of A). Notice that based on the arrows, the elements of 2^A are grouped into four distinct clusters. Within each cluster, all of the elements relate to each other (including themselves, because R is reflexive), and if two elements are in different clusters, then neither relates to the other.

Similarly we can construct such clusters for the examples under the definition of transitivity. In example 1, every real number belongs to one cluster. In example 2, we can construct 13 clusters, and each number p is in the $(p \bmod 13)$-th cluster.

In general, every equivalence relation on a set S induces a partition of S, and each subset (also known as "block") of the partition is known as an equivalence class. In each equivalence class, all of the elements relate to each other (including themselves).

Example 1: Consider the equivalence relation $R = \{(x, y) : 0 \leq x, y \leq 1 \text{ or } x = y\}$ on $\mathbb{R}$. One equivalence class is the interval $[0, 1]$. The remaining equivalence classes are all of the form $\{x\}$ for every $x < 0$ and $x > 1$.

Example 2: Consider the equivalence relation $R = \{(x, y) : x - y \bmod 3 = 0\}$ on $\mathbb{Z}^+$. There are three equivalence classes of R: $\{x \in \mathbb{Z}^+ : x \bmod 3 = 0\}, \{x \in \mathbb{Z}^+ : x \bmod 3 = 1\}$, and $\{x \in \mathbb{Z}^+ : x \bmod 3 = 2\}$

Example 3: Let X be a finite set, and let S be the set of all relations on X. Consider the equivalence relation $R = \{(r_1, r_2) : \text{ range}(r_1) = \text{ range}(r_2)\}$ on S. Let us compute the number of equivalence classes in R. Notice that two relations are in the same equivalence class if they have the same range, so the number of classes is the number of distinct ranges among all relations on S. Since any subset of X is a potential range, the number of equivalence classes in R is $2^{|X|}$.

Example 4: Consider the same setting as Example 3, but let $R = \{(r_1, r_2) : |\text{range}(r_1)| = |\text{range}(r_2)|\}$. Now, the number of equivalence classes is the number of different sizes of the ranges of the relations in S, which is the number of different sizes of the subsets of X. The size of any subset of X is in the set $\{0, 1, \ldots, |X|\}$, so there are $|X| + 1$ equivalences classes.

2.2 Partial Orders

2.2.1 Partial Orders

Recall that a relation on a set satisfying reflexivity, symmetry, and transitivity is known as an equivalence relation. We now introduce another type of relation by first defining a new property.

Definition 15. *A relation R on a set S is* asymmetric *if, for every $a, b, \in S$, $(a, b) \in R$ implies $(b, a) \notin R$.*

Definition 16. *A relation is a* partial order *if it is transitive and asymmetric.*

Example 1: Let S be some non-empty set, and 2^S be the power set of S. Let us show that the relation $R = \{(X, Y) : X \subset Y\}$ on S is a partial order.

- Transitive: If $(X, Y) \in R$ and $(Y, Z) \in R$, then $X \subset Y \subset Z$ so $X \subset Z$, which means $(X, Z) \in R$.
- Asymmetric: If $(X, Y) \in R$, then $X \subset Y$, so $Y \not\subset X$. Therefore, $(Y, X) \notin R$.

Note that in a partial order, it is not necessarily the case that any two elements must relate to each other in one way or another. For example, in the relation defined above, two disjoint sets do not relate to each other at all. However, if we strengthen the notion of ordering, then we can obtain a relation that "orders" *every* pair of elements in the underlying set. This is formalized by the following definition.

Definition 17. *Let R be a partial order on a set S. We say that R is a* total order *if it satisfies the following: for every distinct elements $a, b \in S$ such that $a \neq b$, either $(a, b) \in R$ or $(b, a) \in R$.*

Observe that if R is a partial order, then we cannot have $(a, b) \in R$ and $(b, a) \in R$ for any $a, b \in S$. Therefore, if R is a total order, this still cannot be the case.

Example 1: Consider the relation $R = \{(x, y) : x < y\}$ defined on $\{1, 2, 3\}$. It is straightforward to verify that R is a partial order. If $x, y \in \{1, 2, 3\}$ and $x \neq y$, then $x < y$ or $y < x$, so R is a total order.

Example 2: Consider the following relation defined on $\mathbb{R}^2$:

$$R = \left\{((x_1, y_1), (x_2, y_2)) : \sqrt{x_1^2 + y_1^2} < \sqrt{x_2^2 + y_2^2}\right\}.$$

We claim R is a partial order but not a total order.

- Asymmetric: If $((x_1, y_1), (x_2, y_2)) \in R$, then $\sqrt{x_1^2 + y_1^2} < \sqrt{x_2^2 + y_2^2}$, so $((x_2, y_2), (x_1, y_1)) \notin R$.
- Transitive: If $((x_1, y_1), (x_2, y_2)) \in R$ and $((x_2, y_2), (x_3, y_3)) \in R$, then $\sqrt{x_1^2 + y_1^2} < \sqrt{x_2^2 + y_2^2} < \sqrt{x_3^2 + y_3^2}$, so $((x_1, y_1), (x_3, y_3)) \in R$.

Now consider the following two elements of $\mathbb{R}^2$: $a = (-1, -1)$ and $b = (1, 1)$. Since $\sqrt{(-1)^2 + (-1)^2} = \sqrt{1^2 + 1^2}$, we have $(a, b) \notin R$ as well as $(b, a) \notin R$. Therefore, R is not a total order.

Example 3: Consider the relation $R = \{((x_1, y_1), (x_2, y_2)) : x_1 < x_2\}$ on $\mathbb{R}^2$. It is easy to verify that R is a partial order. However, the elements $(1, 1)$ and $(1, 0)$ do not relate to each other in either direction, so R is not a total order.

Example 4: Let F be the set of all functions on a finite set X. Let R be a relation defined on F as follows: $R = \{(f_1, f_2) : \text{Range}(f_1) \subset \text{Range}(f_2)\}$. Again, we claim R is a partial order but not a total order.

- Asymmetric: If $(f_1, f_2) \in R$, then $\text{Range}(f_2) \supset \text{Range}(f_1)$, so $(f_2, f_1) \notin R$.
- Transitive: If $(f_1, f_2), (f_2, f_3) \in R$, then $\text{Range}(f_1) \subset \text{Range}(f_2) \subset \text{Range}(f_3)$ and thus $(f_1, f_3) \subset R$.

Now consider two functions f_1, f_2 with same range. Then neither range is a proper subset of the other, so we have $(f_1, f_2) \notin R$ and $(f_2, f_1) \notin R$, so R is not a total order.

Example 5: Consider two rational numbers $r_1, r_2 \in \mathbb{Q}$, expressed in lowest terms as $r_1 = p_1/q_1$ and $r_2 = p_2/q_2$. We define R as a relation on $\mathbb{Q}$ as follows: $(r_1, r_2) \in R$ if and only if one of the following holds:

1. $p_1 + q_1 < p_2 + q_2$

2. $p_1 + q_1 = p_2 + q_2$ and $p_1 < p_2$.

We claim that R is a total order.

- Asymmetric: Suppose $(r_1, r_2) \in R$. If $p_1 + q_1 < p_2 + q_2$, then $(r_2, r_1) \notin R$. Similarly, if $p_1 + q_1 = p_2 + q_2$ and $p_1 < p_2$, then $(r_2, r_1) \notin R$.

- Transitive: Suppose $(r_1, r_2), (r_2, r_3) \in R$. Then consider the following cases:

  Case 1. $p_1 + q_1 < p_2 + q_2$: Notice $p_2 + q_2 \leq p_3 + q_3$, so $p_1 + q_1 < p_3 + q_3$, so $(r_1, r_3) \in R$.

  Case 2. $p_1 + q_1 = p_2 + q_2$ and $p_1 < p_2$

  Case 2(a). $p_2 + q_2 < p_3 + q_3$. Then $p_1 + q_1 = p_2 + q_2 < p_3 + q_3$, and thus $(r_1, r_3) \in R$.

  Case 2(b). $p_2 + q_2 = p_3 + q_3$ and $p_2 < p_3$. Then $p_1 + q_1 = p_2 + q_2 = p_3 + q_3$ and $p_1 < p_2 < p_3$. Therefore, $(r_1, r_3) \in R$.

Now we show that R is a total order. Let $r_1, r_2 \in \mathbb{Q}$ be distinct rationals and suppose $(r_1, r_2) \notin R$. There are two ways this can happen:

1. $p_1 + q_1 > p_2 + q_2$: In this case, we have $(r_2, r_1) \in R$.

2. $p_1 + q_1 = p_2 + q_2$ and $p_1 \geq p_2$: Notice that if $p_1 = p_2$, then we would have $q_1 = q_2$, so $r_1 = r_2$, contradicting our assumption that r_1 and r_2 are distinct. Thus, $p_2 + q_2 = p_1 + q_1$ and $p_2 < p_1$, which implies $(r_2, r_1) \in R$.

In both cases, we have $(r_2, r_1) \in R$, as desired.