COMPSCI 230: Discrete Mathematics for Computer Science 3/24/2020

Recitation 11: Graph Theory (directed graphs, MSTs, cycles)

Created By: David Fischer; adapted from Erin Taylor, Kevin Sun, Alex Steiger

1. Recall that it is not necessary that a graph be connected. Let G be an acyclic graph, c represent the number of connected components, n the number of vertices and m the number of edges. Prove that $m = n - c$ (note that as a consequence of this, $m = n - 1$ in trees).

   (a) *Proof.* We proceed by induction on m, the number of edges.
   Base case is $m = 0$, so that every vertex is its own connected component, and so $c = n$, and so $0 = 0 \to 0 = n - c$ as desired.
   Suppose for all graphs with k edges we have $k = n - c$. Consider an acyclic graph $G = (V, E)$ with $k + 1$ edges. Remove one edge to get $G' = (V, E')$. Note that since G is acyclic, removing this edge increases the number of connected components by one. Then, we know $k = n - c' = n - c - 1 \to k + 1 = n - c$, as desired. □

2. We have seen three properties of trees: 1) number of edges = number of vertices - 1, 2) acyclic, and 3) connected. It is true that any two of these properties implies the third, so a valid definition of a tree need only include two of the properties. Prove that $1 \wedge 2 \to 3$, $2 \wedge 3 \to 1$ and $1 \wedge 3 \to 2$.

   (a) $2 \wedge 3 \to 1$

   Proof. This is the result from problem one with $c = 1$. □

   (b) $1 \wedge 2 \to 3$

   Proof. Suppose we are given a graph in which $|E| = m = n - 1 = |V| - 1$ and which is acyclic. We will show this graph is connected. For the sake of contradiction, suppose it is not connected so that c as defined above is 2. By the theorem from problem 1, this means $m = n - 2$, which contradicts our definition of the graph. □

   (c) $1 \wedge 3 \to 2$

   Proof. Suppose we are given a graph such that $m = n - 1$, and that is connected. We will see that it is acyclic. Suppose not, so suppose there is a cycle. Remove some edge in this cycle, and call the new graph G', with $m' = m - 1$ edges and n vertices. However, this means $m' = m - 1 = n - 2$, which contradicts the fact that this since graph is still connected, we must have from problem one $m' = n - 1$. □

3. We have seen the concept of minimum spanning trees. We have seen two properties of MSTs: the cycle property and the cut property. Give proofs for these.

   (a) Cycle Property : A largest edge in every cycle will not be in a MST.

Proof. Suppose note. Then there is a graph G with an MST T such that T contains the largest edge $e = (v_1, v_2)$ from a cycle C of G. Since e is part of C, there is a path $P = \{(v_2, v_3), ..., (v_k, v_1)\}$ to get from v_2 to v_1. Now note that $c(v_1, v_2) >= c(v_2, v_3)$ since e is a largest edge in the cycle. So, we can remove e from T and add (v_2, v_3) to T to get T'. T' is a tree since it has the same number of edges as T and is connected. As well, the cost of T' is at most the cost of T. In either case T' is a MST, and so there is a MST that does not include a (the) largest edge e. □

(b) Cut Property: A smallest edge in every cut will be in a MST.

Proof. Consider a graph $G = (V, E)$, and some cut $A \subset V$ and $B \subset V$ such that $A \cap B = \emptyset$. Consider the set of edges $C = \{(u, v) | u \in A \wedge v \in B\}$, and the lightest edge $e = (u_1, u_2) \in C$. Now, suppose our claim were not true, and all MSTs did not include e. Let $T = (V, E')$ represent such a MST. Note that since $e \notin E'$, $\exists e' \in C$ such that $e' \in E'$. By definition of e, $c(e) \leq c(e')$. Therefore, we can consider $T' = (V, E'')$ where $E'' = (E' \setminus \{e'\}) \cup \{e\}$. Note that $c(T'') \leq c(T)$, and, since $e \in C$, the graph is connected and $|E''| = |E'| = |V| - 1$ so that T'' is a valid MST. This contradicts our assumption that no MST included e. □

4. You want to go to your friends house, but it is really hot outside, and you don't have a car (you'll have to walk). You know which streets in your town are shady. Consider the problem of getting to your friends house while passing *only* on shady streets. What problems does solving this problem amount to from a graph theoretic standpoint? Bonus, suppose it is not possible to get to your friends house by traveling only on shady roads. Write the problem of getting to your friends house while going on the fewest number of non shady roads (as a minimization problem).

   (a) Consider creating a graph of roads, where every intersection is a vertex, and there is an edge between two intersections iff there is a single stretch of road with no intersections between them. Remove all edges which are not shady. If you and your friend are still connected, then there is a way to get to your friend's house without braving a direct hit by the sunlight.

5. Suppose you are an Internet Service Provider, and are expanding service to a new city. It costs some amount to connect a building to every other building, which you know thanks to hard work done by your research team. You have purchased some network hubs, and want to connect every building in the city to your service using the least amount of money possible. How might you model this problem from a graph theoretic standpoint?

   (a) Consider the undirected, weighted, complete graph where each node is a building, and your network hubs are merged into a single vertex. Let the weight of each edge (v_1, v_2) be the cost of building a connection from the building v_1 to v_2 or vice versa. Then, finding the MST is equivalent to finding the cheapest way to connect the whole city to your system.

6. In graph theory, a *hamiltonian path* is a walk that visits every vertex exactly once. A *hamiltonian cycle* is a cycle such that every vertex is visited exactly once (i.e. a hamiltonian path that has an edge from the last vertex to the start vertex). In chess, recall that a knight can move two squares forward and one square to the side. A *knight's tour* is a sequence of moves that a knight can make that visits each square exactly once. We call a knight's tour *reentrant* if there is a sequence of moves that a knight can make that visits each square exactly once, and ends on a square that can return to the square on which the knight started.

(a) Given some m by n grid, describe a graph that models this system.

- Let each square in the grid be a vertex $v_{i,j}$ where $i \in [1, m]$ and $j \in [1, n]$. Let there be a directed edge from $u_{w,x}$ to $v_{y,z}$ if it is possible to reach $v_{y,z}$ by a valid knight's move from $u_{w,x}$ (i.e. $(y = w \pm 2$ and $z = x \pm 1) \vee (y = w \pm 1$ and $z = x \pm 2)$).

(b) Prove that in some m by n grid, there is a reentrant knight's tour if and only if there is some hamiltonian circuit in the graph you constructed.

- Suppose there is a reentrant knight's tour in the board. Then, this corresponds to some path in the graph by definition of the graph. Then, since each square is hit exactly once, each vertex must be hit exactly once. Finally, the last step from the end of the path to the beginning of the path must also be an edge in the graph. Suppose there is some hamiltonian circuit in the graph. Then, since each edge corresponds to a legal knight move, this corresponds to a sequence of moves on the board. Since each graph vertex is present in the circuit exactly once, and the last vertex contains an edge to the start vertex, each square in the grid is hit exactly once, and the there is a legal move from the last square to the first square, and so there is a reentrant knights tour on the board.

(c) Show that a bipartite graph with an odd number of vertices does not have a hamiltonian circuit.

- Suppose there were a graph with an odd number of vertices and a hamiltonian circuit, $C = \{v_0, v_1, ..., v_n\}$ where n is even. By definition of bipartite, each v_{i+1} is in a different partition than v_i, which means every even v_j is in the same bipartition and every odd v_h is in the other bipartition. So, v_0 and v_n must be in the same bipartition, and so there cannot be an edge between them, which means the walk is not a circuit, a contradiction.

(d) Use the previous two results to show that if m and n are both odd, then there is no reentrant knight's tour on an m by n board.

- We will begin by showing the knight's walk graph is a bipartite graph, and then, since it has an odd number of vertices, it cannot contain a hamiltoninan circuit, which means it cannot have a knight's tour. To show the knight's graph is bipartite, it suffices to show it is 2-colorable. We can start at any vertex, color it one color and color all of its neighbors the other color, and so on. We will see this is a valid coloring. If it is not, then there is a node whose neighbors have an edge between them. In other words, there is a node u which has two neighbors v_1 and v_2 such that $(v_1, v_2) \in E$. We will see that this is not possible. Represent u by its x and y coordinates in the grid ($u = (x, y)$). Then let the neighbors of u be the set $N(u) = \{a, b, c, d, e, f, g, h\} = \{(x+1, y+2), (x+1, y-2), (x-1, y+2), (x-1, y-2), (x+2, y+1), (x+2, y-1), (x-2, y+1), (x-2, y-1)\}$. Note that WLOG the parity of both coordinates of u are the same, and even, and the parity of the coordinates of neighbors of u are different, one even and one odd. Further, consider an extension of this argument and let v_1 be a neighbor of u, so that the parity of its coordinates are different (one even and one odd). Then. not that all of its neighbors will have the same parity for both coordinates, even and even or odd and odd. Therefore, none of v_1's neighbors can be a neighbor of u, and so there is no edge between neighbors of a vertex. Thus, the described two-coloring is valid, and so the graph is bipartite. To finish, m and n are both odd, so there is an odd number of vertices in the graph, and since the graph is bipartite with an odd number of vertices, by part c, there is no hamiltonian cycle, and so no reentrant knight's circuit.

7. Suppose you just started a new round of fantasy baseball, and it is time to select new players for your dream team wish list. There are 9 roles on your team, and each player has the potential to play a variety of roles. However, once chosen for a role, they are locked in and cannot play any different role. You have devised a system which allows you to assign to each player a certain rating for that player at playing a particular role. Devise a graph theoretic way of describing your attempt to come up with the best possible dream roster.

   (a) Create a bipartite graph where one side is the roles, and the other is the players. Let there be a weighted edge between a player and a role if that player can play that role, with a weight of the numerical value your system created. Then, you want to create a role-perfect matching with maximum weight.

   (b) To build off of this problem a little bit, suppose you know there is a computer program that can download which will solve the *balanced assignment problem* in polynomial time (read: some reasonable amount of time). The balanced assignment problem asks the following, given a bipartite graph with equal sized bipartitions A and B, define a bijective function $f : A \to B$ with minimum weight (i.e. $\min\left(\sum_v c((v, f(v)))\right)$ where c is the cost function). Describe how you could change your definition of the graph so you could use this program to give you the assignment you want.

   (c) *solution:* First, negate each edge weight. If A is the set of players and B the set of team roles, add to A a node for each role and add to B a node for each player, and similarly add an edge between a player and role with a cost equal to the negated value that your system gives you for the player working at that role. We now have a graph that works with the program given, and we can find our optimal team!

8. Consider a set of airplanes waiting to land at an airport. Some airplanes have more gas than others, and it is important that airplanes with less gas land before airplanes that have enough fuel to wait. Consider how to model this system using a graph. The graph constructed should always lead to a way to order the airplanes so that an airplane with less fuel lands before an airplane with more fuel. Prove that such an ordering exists.

   (a) Consider the directed graph where each vertex is a plane, and there is a directed edge from plane 1 to plane 2 if plane 2 has more fuel than plane 1. If two planes have the same amount of fuel there is no edge between them in either direction. The result is a directed, acyclic graph (DAG), and so a topological ordering exists. Suppose there were a cycle in the graph. Then, there is a walk $w = (v_1, ..., v_n, v_1)$. By the transitive property of the relation "has more fuel than", this implies the amount of fuel v_1 has is strictly greater than the amount of fuel v_1 has, a contradiction.

9. Formulate Sudoku in graph theoretic terms

   (a) Let each square in the Sudoku grid be a vertex. Let there be an undirected edge between two vertices if they are in the same row, column, or block of the grid. Then, solving the Sudoku puzzle corresponds to finding a valid coloring of size 9 in the resulting graph. Usually, we are given a few colors, and are asked to fill in the rest, so it is more accurately a precolor extension problem, rather than a straight coloring problem. Although trivial, we formally state why a solution to the Sudoku puzzle corresponds to a valid 9-coloring. A solution to the puzzle has the property that each number appears exactly once in each row, block and column. Another way of saying this is that if you are given two numbers that are the same, they must be in different rows, blocks and columns, and so the squares they are in must not be connected by an edge in the graph, and it is valid to color them the same color. A solution to a Sudoku puzzle has every square filled in, so every vertex in the graph is assigned a color. Therefore, the coloring given by assigning each number a color and coloring every vertex corresponding to a square containing that number that color is valid. Similarly for the other direction, given a valid nine coloring, we can assign each color a number (perhaps according to some colors and numbers we were given to start), and by definition of the graph, filling the grid with the appropriate numbers is a valid solution.