COMPSCI 230: Discrete Mathematics for Computer Science

Recitation 2: Practice with Propositions, Predicates

Spring 2020 *Created By: David Fischer*

1 De Morgan's Laws

A fun aside - in English grammar, we actually have a special way of denoting some of De Morgan's laws. If we say ″It is not this or that″, formally speaking we are actually saying ″it is not (this), or it is that″. If we speak correctly, though, and say ″It is neither this nor that,″ we mean, ″it is not this and it is not that″. In other words, ″not either (A or B)″ is the same as ″neither A nor B″ (which is proper English grammar), which is the same as (not A) and (not B).

1.1 Show that the following statements are equivalent.

(a) **It is neither raining outside, nor is it sunny outside. It is not raining outside and it is not sunny outside.**

(b) **A positive integer, x, cannot be both even and odd is equivalent to saying x is either not even, or x is not odd.**

1.2 Recall the Venn Diagram method of visualizing de Morgan's laws. Go over set notation which says the same thing - union of complements is the complement of the intersection.

1.3 Write down an equivalent statement for each of the following.

(a) **A cat in a closed box is not both alive and dead.** A cat in a closed box is either not dead, or not alive.

(b) **The season cannot be both winter and summer.** The season must be either not winter or not summer.

(c) **The swimmer is neither running nor jumping.** The swimmer is not running and is not jumping.

2 Propositional Algebra

2.1 Write the following formula with only one negation: $\neg(\neg A \vee \neg B \vee C)$

The two steps are to pull out the negation for neg A or neg B term and then distribute the outer negation.

2.2 Show that de Morgan's Law holds for 3 (or more) variables using laws of propositional algebra (including de Morgan's laws on two variables).

$$\begin{aligned}\neg(A \wedge B \wedge C) &\equiv \neg((A \wedge B) \wedge C) \\ &\equiv \neg(A \wedge B) \vee \neg C \\ &\equiv \neg A \vee \neg B \vee \neg C\end{aligned}$$

2.2.1 Order of Operations

There is a quick tangent here on formal order of operations for logical operators. Formally speaking, the sentence above is an invalid sentence since it is unclear which $\wedge$ operation to do first. In this case, it doesn't matter since $\wedge$ is associative, but it does make a difference once you include propositional statements with more than one operator. Since there is no formal convention for order of operations on logical operators, we are free to define our own conventions for ambiguities. For example, in this class we will use the following ordering:

$$\begin{aligned} &() \\ &\neg \\ &\wedge \\ &\vee \\ \rightarrow &\leftarrow \\ &\leftrightarrow \end{aligned}$$

In most situations we will use parentheses to avoid ambiguous statements.

2.3 Show the following formulas are equivalent using only laws of propositional algebra: $(P \wedge \neg Q) \vee (\neg P \wedge Q)$, and $(\neg Q \vee \neg P) \wedge (P \vee Q)$.

$$\begin{aligned}(P \wedge \neg Q) \vee (\neg P \wedge Q) &\equiv \neg(\neg P \vee Q) \vee \neg(P \vee \neg Q) \\ &\equiv \neg((\neg P \vee Q) \wedge (P \vee \neg Q)) \\ \text{Now, we distribute} & \\ &\equiv \neg \left[((\neg P \vee Q) \wedge P) \vee ((\neg P \vee Q) \wedge \neg Q)\right] \\ &\equiv \neg \left[((\neg P \wedge P) \vee (Q \wedge P)) \vee ((\neg P \wedge \neg Q) \vee (Q \wedge \neg Q))\right] \\ \text{Note that } \neg P \wedge P = \textsc{False}\text{, and similarly for } Q & \\ &\equiv \neg \left[(Q \wedge P) \vee (\neg P \wedge \neg Q)\right] \\ &\equiv \neg(Q \wedge P) \wedge \neg(\neg P \wedge \neg Q) \\ &\equiv (\neg Q \vee \neg P) \wedge (P \vee Q)\end{aligned}$$

This is the final result we want. Does this look familiar to anyone? The first proposition says one of the two literals must be false, and the second says that one of them must be true... $\oplus$ (if you need, you can verify with a truth table).

2.4 Write the CNF and DNF representations of the following formulas.

(a) $(A \land B) \lor \neg C$ We first draw the truth table.

A	B	C	$A \land B$	$\neg C$	$(A \land B) \lor \neg C$
TRUE	TRUE	TRUE	TRUE	FALSE	TRUE
TRUE	TRUE	FALSE	TRUE	TRUE	TRUE
TRUE	FALSE	TRUE	FALSE	FALSE	FALSE
TRUE	FALSE	FALSE	FALSE	TRUE	TRUE
FALSE	TRUE	TRUE	FALSE	FALSE	FALSE
FALSE	TRUE	FALSE	FALSE	TRUE	TRUE
FALSE	FALSE	TRUE	FALSE	FALSE	FALSE
FALSE	FALSE	FALSE	FALSE	TRUE	TRUE

Now we consider the rows which result in a true statement. Each assignment is an and of literals. If we have even one of those assignments, the result is TRUE, so we can take the $\lor$ of those assignments to get the DNF formula as follows. $(A \land B \land C) \lor (A \land B \land \neg C) \lor (A \land \neg B \land \neg C) \lor (\neg A \land B \land \neg C) \lor (\neg A \land \neg B \land \neg C)$. To the get CNF is slightly more complicated. We want to not have any of the FALSE outcomes. So, we take the $\land$ of the negation of all assignments that make the outcome false, as follows. $\neg(A \land \neg B \land C) \land \neg(\neg A \land B \land C) \land \neg(\neg A \land \neg B \land C)$. The final step to get the CNF is to apply de Morgan's laws.

(b) This is a less trivial example. $A \land \neg((B \lor \neg C) \land (\neg A \lor C)) \land (B \lor C)$. The truth table follows.

A	B	C	$B \lor C$	$B \lor \neg C$	$\neg A \lor C$	$\neg((B \lor \neg C) \land (\neg A \lor C))$	$A \land \neg((B \lor \neg C) \land (\neg A \lor C))$	$A \land \neg((B \lor \neg C) \land (\neg A \lor C)) \land (B \lor C)$
TRUE	TRUE	TRUE	TRUE	TRUE	TRUE	FALSE	FALSE	FALSE
TRUE	TRUE	FALSE	TRUE	TRUE	FALSE	TRUE	TRUE	TRUE
TRUE	FALSE	TRUE	TRUE	FALSE	TRUE	TRUE	TRUE	TRUE
TRUE	FALSE	FALSE	FALSE	TRUE	FALSE	TRUE	TRUE	FALSE
FALSE	TRUE	TRUE	TRUE	TRUE	TRUE	FALSE	FALSE	FALSE
FALSE	TRUE	FALSE	TRUE	TRUE	TRUE	FALSE	FALSE	FALSE
FALSE	FALSE	TRUE	TRUE	FALSE	TRUE	TRUE	TRUE	FALSE
FALSE	FALSE	FALSE	FALSE	TRUE	FALSE	TRUE	FALSE	FALSE

Note that we can save time in filling out the table - once we know A is false, we know the result is false, for example. Please let me know if you find errors in the table! Anyway, taking the rows that are true, and saying the whole formula is true if the truth assignment to the literals is any of the truth assignments that lead to that gives the following DNF: $(A \land B \land \neg C) \lor (A \land \neg B \land C)$. The CNF is more tricky. We say, if the assignment is one that does not correspond to the first row, and does not correspond to the fourth row, and does not correspond to the fifth row, etc, then the formula is true gives the following CNF:

$$\neg(A \wedge B \wedge C) \wedge \neg(A \wedge \neg B \wedge \neg C) \wedge \neg(\neg A \wedge B \wedge C) \wedge \neg(\neg A \wedge B \wedge \neg C) \wedge \neg(\neg A \wedge \neg B \wedge C) \wedge \neg(\neg A \wedge \neg B \wedge \neg C)$$

In fact, we could simplify the last four terms to simply $\wedge A$, but we don't require simplification for homework answers, for example.

This gives an easy way to determine if two formulas are equivalent; namely, put them into CNF or DNF form using this mechanical process, and then see if they are the same. Note that the specific order of the terms depends on how you set up the truth values of the literals!

3 Contrapositive

3.1 Prove the validity of the contrapositive using properties of propositional logic.

This proof is quite simple. and outlined below.

3.1.1 Write the following formulas using only $\neg$, $\wedge$ and $\vee$.

(a) $A \to B$

$$\neg A \vee B$$

(b) $\neg B \to \neg A$

$$B \vee \neg A$$

3.1.2 Using the previous two results, derive the validity of the contrapositive using properties of propositional operators.

The commutative law applied to the previous answers gives what is desired.

3.2 Write the following propositions as predicate formulas and give the contrapositive statements.

(a) **For a positive integer x, if $3x + 2$ is odd, then x is odd.**
First, we write the statement itself as a predicate formula. Let $f(x) = 3x + 2$. Let $g(x) =$ "x is odd". Our predicate formula should then be

$$\forall x \in \mathbb{Z},\ g(f(x)) \to g(x)$$

The contrapositive of the formula is then

$$\begin{aligned} \forall x \in \mathbb{Z},\ \neg g(x) &\to \neg g(f(x)) \\ &\equiv g(x) \vee \neg g(f(x)) \end{aligned}$$

(b) **All integers that are even and square are divisible by 4.**
Let $f(x) = "x \text{ is even}"$. Let $g(x) = "x \text{ is square}"$. Let $h(x) = "x \text{ is divisible by } 4"$. Then our formula should be as follows

$$\begin{aligned} &\forall x \in \mathbb{Z}, f(x) \wedge g(x) \rightarrow h(x) \\ \equiv &\forall x \in \mathbb{Z}, \neg(f(x) \wedge g(x)) \vee h(x) \end{aligned}$$

The contrapositive is then

$$\begin{aligned} &\forall x \in \mathbb{Z},\ \neg h(x) \rightarrow \neg(f(x) \wedge g(x)) \\ \equiv &\forall x \in \mathbb{Z},\ h(x) \vee (\neg f(x) \vee \neg g(x)) \end{aligned}$$

(c) **All integers greater than 8 can be written as a sum of 3s and 5s.**
Let $x \in \mathbb{Z}$, and $f(x) = "x \text{ is greater than } 8"$. Let $g(x, y) = 3 * x + 5 * y$, and define $h(x, y) \leftrightarrow (x = y)$. Then our formula takes the form as follows

$$\begin{aligned} &\forall x \in \mathbb{Z},\ \exists a, b \in \mathbb{Z} \,.\, f(x) \rightarrow h(x, g(a, b)) \\ \equiv &\forall x \in \mathbb{Z}, \exists a, b \in \mathbb{Z} \,.\, \neg f(x) \vee h(x, g(a, b)) \end{aligned}$$

The contrapositive is then

$$\begin{aligned} \forall x \in \mathbb{Z},\ \exists a, b \in \mathbb{Z} \,.\, \neg h(x, g(a, b)) \rightarrow \neg f(x) \\ \equiv h(x, g(a, b)) \vee \neg f(x) \end{aligned}$$

4 Goldbach's Conjecture

4.1 Goldbach's conjecture in english is as follows.

Every even number greater than 2 can be written as the sum of two primes.

4.2 What is the predicate formula for Goldbach's Conjecture?

Let $f(x) = “x \text{ is even and greater than } 2”$. Let $g(x) = "x \text{ is prime}"$. Let $h(x, y, z) \leftrightarrow (x = y + z)$.

$$\forall x \in \mathbb{Z},\ \exists y, z \in \mathbb{Z}.\, g(y) \wedge g(z) \wedge f(x) \rightarrow h(x, y, z)$$

Alternatively, we can define a set P of prime numbers, a set E of even numbers strictly greater than two, and keep h as defined above. This allows us to say the following.

$$\forall x \in E,\ \exists p, q \in P \,.\, h(x, p, q)$$

4.2.1 The order of quantifiers matters. What does the following predicate formula say?

$$\exists y, z \in \mathbb{Z}, \forall x \in \mathbb{Z} \,.\, g(y) \wedge g(z) \wedge f(x) \rightarrow h(x, y, z)$$

This says that there are two prime numbers, such that every even integer greater than two can be received as the output of adding the two prime numbers together. This is clearly false! Two prime numbers can only ever equal one number. Now you might think this is specific to Goldbach's conjecture, and is maybe tied to the fact that we don't have a proof for it. In fact, the order of quantifiers matters in general, and you should try it yourself on some of your favorite formulas!