Recitation 7: Set Basics

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1. Set Notations

- $A^C \equiv \overline{A} \equiv A'$ - the complement of A, that is the set of all elements in the universe but not in A
- $\in$ - "in", element-wise. "Z" $\in A$ means "Z" is an element of A.
- $\notin$ - "not in" as above. "Z" $\notin A \equiv$ "Z" $\in \overline{A}$
- $|A|$ - the "cardinality" of A, ie the number of elements in A
- $A \subseteq B$ - every element in A is in B; A may be equal to B
- $A \subset B$ - every element in A is in B but $A \neq B$
- $\varnothing \equiv \{\}$ - the empty set, with cardinality zero

We've already been using sets rather extensively in this class - as the domains of discourse for predicates and quantifiers! A set is an unordered collection of items, drawn from some (probably larger) universe of items. This points to two different ways of defining a set: the first is to explicitly write out all elements in it, eg. $A = \{1, 2, "B", "Xerox", 59, \{\varnothing\}, "!"\}$. We say the elements of a set are "in" the set, which is denoted as "!" $\in A$. We can also say that elements are not in a set, which is simply denoted "?" $\notin A$. Note that it is not required there be some unifying theme among the elements of the set, as long as we take the *universe* of possible items to be large enough. That said, this method of describing sets runs into problems! What if we want to describe an infinite set, like the set of rational numbers $\mathbb{Q}$? Then we can define a predicate on a variable, and all things that satisfy that predicate are in the set. Often, we have an implicit universe in mind when considering a set. This is the larger set under consideration, given the context in which the argument is being placed. We often talk about the *complement* of a set, which consists of all elements in the universe not in the set. The notation for complement can be A', $\overline{A}$, or A^c. Note that if something is in the universe, but not in the set, we can say it is in the complement of the set, or "?" $\in \overline{A}$. To further tack on terminology, we call the number of elements of a set the *cardinality* of the set, and denote it $|A|$. Finally we will give notions of sets containing other sets (not as an element!) as follows. We say that A is a *subset* of B, denoted $A \subseteq B$, if every element A is also in B. If A is not equal to B we say that A is a *proper subset* of B, denoted $A \subset B$ if every element in A is also in B but $A \neq B$ (ie $\exists b \in B.b \notin A$). Note that in general we can treat sets themselves as elements in another set. We will get familiar with these concepts using examples

- In the universe of all fish, what is the complement of the set of saltwater sharks?
  ans: the set of all fish which are either not saltwater fish, or are not sharks

- In the universe defined as the set of all snow days, what is the complement of the set of snow days during which wither school is cancelled or you have to attend school?
  ans: the set of snow days in which school is not cancelled and you do not have to attend school

- In the universe defined as
  $U = \{53, 2, \text{the set of all types of chairs}, \text{the set of all sets with size equal to the number of chair types}\}$, what is the complement of the set of all sets with size equal to the number of chair types?
  ans: $\{53, 2, \text{the set of all types of chairs}\}$ - note here that the set of all types of chairs has size equal to the number of chair types, but that just indicates that it is an element of the set we take the complement of - it is still independently in the universe.

- In the universe of real numbers, what is the complement of the set of rational numbers?
  the set of irrational numbers

- What is $|A|$ where $A = \{x \in \mathbb{N}. 0 < x < 10\}$? *ans: 9*

- Is $\{\sqrt{2}, \sqrt{3}\} \in \mathbb{R}$? *ans: no. The element $\sqrt{2} \in \mathbb{R}$ as is $\sqrt{3}$, but the set containing both is not an element of the reals.*

- What is the cardinality of the set containing the empty set ($|\{\emptyset\}|$)? *ans: 1 (not 0)*

Here we introduce more notations and definitions of binary operators on sets. We define the *intersection* of two sets A and B, written as $A \cap B$, to be the set of all elements that are in A, and in B. We further define the *union* of A and B, written $A \cup B$ as the set of all elements either in A, or in B. We also define *set difference*, denoted $A \setminus B$, which is the set of all items in A that are not in B. Terminology for set difference includes the *relative complement* with respect to A in B. We say two sets A and B are disjoint if they have no elements in common, i.e. $A \cap B = \emptyset$. There are various ways of writing these operations, and the venn diagrams for the operations should be drawn for reference.

- $A \setminus B$ - set minus: the set of all elements in A not in B

- $A \cup B$ - union: the set of all elements either in B or in A

- $A \cap B$ - intersection: the set of all elements in A and in B

We explore these concepts in the following examples:

- Is $\cap$ associative? Is $\cup$?
  ans: yes to both. Proof by taking arbitrary element. Venn diagram gives intuition

- Is $\cap$ commutative? Is $\cap$?
  ans: yes to both.

- Does $(A \cap B) \cup C = A \cap (B \cup C)$?
  ans: no

- Demorgan's laws for $\cup$. What is $\overline{A \cup B}$?
  ans: $\overline{A} \cap \overline{B}$

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- Express $A \cap B$ in terms of $\cup$ and complement.
  ans: $A \cap B = \overline{\overline{A} \cup \overline{B}}$

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- Express $A \cap B$ in terms of set difference.
  ans: $A \cap B = A \setminus (A \setminus B)$

- How can we write set minus using only intersection and complement?
  ans: $A \setminus B = A \cap \overline{B}$

- Prove or disprove: $A \setminus B \subseteq A$.
  Given $x \in A \setminus B$, $x \in A$ by definition. Therefore every element of $A \setminus B$ is in A.

- Prove or disprove: $(A \setminus B) \cap C = (A \cap C) \setminus (B \cap C)$.
  ans:

$$
\begin{aligned}
(A \cap C) \cap (\overline{B \cap C}) &= (A \cap C) \cap (\overline{B} \cup \overline{C}) && \text{(definition of set difference, de Morgan's law)} \\
&= (A \cap C \cap \overline{B}) \cup (A \cap C \cap \overline{C}) && \text{(left distribution of $\cap$ over $\cup$)} \\
&= (A \cap C \cap \overline{B}) \cup (A \cap \emptyset) && \text{(definition of complement)} \\
&= (A \cap \overline{B}) \cap C && \text{(domination of $\emptyset$ in $\cap$, commutativity and associativity)} \\
&= (A \setminus B) \cap C && \text{(definition of set difference)}
\end{aligned}
$$

- Prove or disprove: $(A \setminus B) \cup C = (A \cup C) \setminus (B \cup C)$
  ans: We disprove by counter example: Consider $A = \{1\}$, $B = \{2\}$, $C = \{3\}$. Then $(A \setminus B) \cup C = \{1,3\}$, but $(A \cup C) \setminus (B \cup C) = \{1,3\} \setminus \{2,3\} = \{1\}$

- What is $\mathbb{Q} \cup \{x | x \text{ is irrational}\}$?
  ans: $\mathbb{R}$ by definition

We take this opportunity to give a little more notation. We define the *powerset* of a set A as being the set of all possible sets of elements of A. For example, if $A = \{1,2,3\}$ $P(A) = \{\{\}, \{1\}, \{2\}, \{3\}, \{1,3\}, \{1,2\}, \{2,3\}, \{1,2,3\}\}$. Note the empty set is always a part of the powerset. Question: how many elements are in the powerset of a set with cardinality n? *ans: 2^n - intuitively, as we build the sets, each element has two options, either in the set or out of it. There are thus 2^n possible outcomes for each set, so the powerset has 2^n sets.*

2. The **symmetric difference** of sets A and B denoted $A \triangle B$ is defined as

$$A \triangle B := (A \setminus B) \cup (B \setminus A)$$

(a) Prove that $A \triangle B = (A \cup B) \setminus (A \cap B)$.

(b) Prove that $\overline{A \cup B}$ is disjoint from $A \triangle B$.

(c) Prove that $A \cap B$ is disjoint from $A \triangle B$.

Answer:

(a) Let $\mathcal{U}$ be the absolute universe containing A and B.

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\begin{aligned}
A \triangle B &= (A \setminus B) \cup (B \setminus A) && \text{(definition of symmetric difference)} \\
&= (A \cap \overline{B}) \cup (B \cap \overline{A}) && \text{(definition of set difference)} \\
&= ((A \cap \overline{B}) \cup B) \cap ((A \cap \overline{B}) \cup \overline{A}) && \text{(left distribution of } \cup \text{ over } \cap) \\
&= ((A \cup B) \cap (\overline{B} \cup B)) \cap ((A \cup \overline{A}) \cap (\overline{A} \cup \overline{B})) && \text{(right distribution of } \cup \text{ over } \cap) \\
&= ((A \cup B) \cap (\mathcal{U})) \cap (\mathcal{U} \cap (\overline{A} \cup \overline{B})) && \text{(definition of complement)} \\
&= (A \cup B) \cap (\overline{A} \cup \overline{B}) && \text{(identity of } \cap) \\
&= (A \cup B) \cap \overline{(A \cap B)} && \text{(de Morgan's law)} \\
&= (A \cup B) \setminus (A \cap B) && \text{(definition of set difference)}
\end{aligned}
$$

Note that steps 5 and 6 might require some explanation.

(b) Consider an arbitrary item in $\overline{A \cup B}$. Note this element cannot be in $A \cup B$. However from (a) we saw that any element in $A \triangle B$ must be an element of $A \cup B$ by the definition of set difference.

(c) Suppose there were an item in $A \cap B$ in $A \triangle B$. Again, this contradicts our definition of set minus and the result from (a). So, there must not be an item in $A \cap B$ also in $A \triangle B$, and so the sets are disjoint.