COMPSCI 230: Discrete Mathematics for Computer Science 2/27/2020

Recitation 8: Relations and Functions

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Relations

Today we are exploring relations and functions on sets. We introduce the notion of relations to generalize functions. To formally recall what a relation is, recall the definition of the Cartesian product of two sets A and B as $A \times B$ is the set of all tuples $(a \quad b)$ such that $a \in A$ and $b \in B$. A relation on two sets A and B defines a subset of the Cartesian product of A and B. More simply, you can think of a relation from A to B as a mapping from the elements of A to the elements in B defined in some arbitrary way. We call A in this case the domain, and B the co-domain. We call all elements of B that are actually capable of being realized (have a element in A which map to them) the range, or the image of the relation. We call all elements in A which actually map to some element in B the support (or preimage). One way to represent a relation is to explicitly write out all the pairs $(a \quad b)$, $a \in A, b \in B$. We can represent a relation as a bipartite graph, with the domain on one side and the codomain on the other side, and arrows from the support to the range as the relation defines. It is extremely helpful to understand these visually. For each of the following sections, draw out diagrams like this:

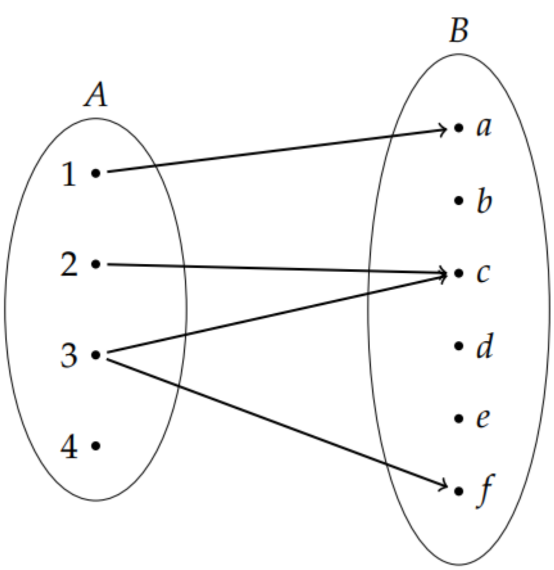

Figure 1: Figure representing relation in Example 2.

- Example: $A = \{a, b, c, d\}$ $B = \{1, 2, 3\}$ and $R : A \to B = \{(a, 1), (a, 2), (b, 1), (c, 2), (d, 1)\}$. What is the domain, codomain, range and support (preimage) of R?
  answer: the domain is A the codomain is B, the range is $\{1, 2\}$ and the support (preimage) is A.

- We can define a relation between the set of people and the set of dogs $O : P \to D$ where $xOy \leftrightarrow$ (x is the owner of y). What is the domain, co domain, range and support (preimage) of O?

- Consider $Q : \mathbb{R} \to \mathbb{R}$ such that $aQb \leftrightarrow (a^2 + b^2 = 25)$. What is the domain, codomain, range, and preimage (or support) of Q?
  answer: domain = $\mathbb{R}$, codomain = $\mathbb{R}$, range = support = $[-5, 5]$

- (20 minutes) In fact, the domain and co domain of a relation need not be clearly related in substance (although there of course needs to be some logical proposition connecting the domain to the co domain). We say that a relation $R : A \to B$ can have several properties (previously defined in class).

  – **Injective:** R is *injective* if and only if all elements of B have in degree at most 1 (≤ 1) - each element on the domain side of the bipartite graph has in-degree at most 1. "vertical line test"

  – **Surjective:** R is *surjective* if and only if all elements of B at in degree at least 1 - each element on the domain side of the bipartite graph has in-degree at least 1. "horizontal line test"

  – **Bijective:** R is *bijective* if and only if all elements of B have in degree exactly 1 - each element on the domain side of the bipartite graph has in-degree exactly 1. R is *bijective* if and only if R is *injective* and *surjective*.

- Define a relation from the set of all cars to the integers $N : \{\text{cars}\} \to \mathbb{Z}$ where $cNx \leftrightarrow c$ has x wheels. Note that this relation defines a subset of $(\{\text{cars}\} \times \mathbb{Z})$. We count motorcycles as cars, so it is clear that the image of the relation is $2, 3, 4, 16$ and maybe some other weird numbers for trucks with multiple carriages or odd custom designed cars. Is N bijective? Injective? Surjective? Why or why not?
  answer: No clearly not, since there is an element of the codomain such that no element of the domain maps to it.

- **Proof Techniques** To show that a relation is injective, it is enough to show that $\forall b \in B.(a, b) \in R \wedge (a', b) \in R \to (a = a')$. To show that a relation is surjective, it is enough to show that $\forall b \in B \exists a \in A.(a, b) \in R$.

- **Functions** Now we define a special property that can be true of relations, and if this property is true of the relation, we call it a *function:* A relation is a function if and only if every element in the domain maps to exactly one element in the range. Note that unlike injective, surjective, bijective, this is a predicate defined on the domain of a relation. Another way of seeing this is that when we look at the bipartite graph of the relation, if every element on the domain side has out degree exactly one, then we call the relation a function. We will see how our familiar, intuitive understanding of functions aligns with the formal definitions that we are learning now.
  IMPORTANT NOTE: This differs from the definition in the text book, which defines a function such that every element in the domain has out degree at most one, and then a *total* function such that every element in the domain has out degree at least one. In this class we use the terminology "function" to describe what the textbook describes as "total function".
  For each of the following, draw the relation on the board.

  – Is the relation O as defined above a function? Injective? Surjective?
    answer: Not a function as one owner can have multiple dogs. Not injective, nor surjective, since a dog can have 0 or 2 owners.

- Is the relation N a function? Injective? Surjective?
  *answer:*If you count a truck as having 16 tires and then 15 if one of them pops, then not a function, but since we can just say that we are defining the relation in terms of the number of tires a car is *supposed* to have, then yes, this is a function. This function is certainly not injective, as many different cars are supposed to have 4 wheels. This function is not surjective, as no car is supposed to have 37 wheels.
- Is the relation $R = \emptyset$ a function? Injective? Surjective?
  answer: By our definition, this relation is not a function, since there are elements in the domain which do not map to anything. This function is injective, since $0 \leq 1$.
- $f(x) = x^2$. Function? Injective? Surjective?
  *answer:*Is a function, but is neither injective nor surjective.
- $x = y^2$. Function? Injective? Surjective?
  answer: Not a function. Injective. Surjective. So, bijective.

- **Proof Techniques**
  To show that a relation is a function, it is sufficient to show two things. First, $\forall a \in A \exists b \in B.(a,b) \in R$. As well, we need that $\forall b \in B(\exists a, a' \in A.(a,b) \in R \wedge (a',b) \in R) \rightarrow a = a'$

- Often, a relation R is defined from a set to itself, as in $R : A \rightarrow A$ in which case we say R is the relation on set A. We are probably very familiar with the *drawing* of functions, where we define functions "on" $\mathbb{R}$, and then draw the subset of $\mathbb{R} \times \mathbb{R}$ with some sort of line.

- Consider $f(x) = x + 5$ is a straight line that crosses the y axes at $y = 5$. Given a sheet of graph paper, we could draw this graph. When we do this, we are implicitly assuming the domain of discourse to be $\mathbb{R}$. To be precise, we could say $f : \mathbb{R} \rightarrow \mathbb{R}$. As a quick exercise, is this function bijective?
  answer: yes. For each element of the codomain, $n \in \mathbb{R}$, there is exactly one element in the domain which maps to it, namely $n - 5$.
  We can further note that this is in fact different from the function $g : \mathbb{Z} \rightarrow \mathbb{R} \leftrightarrow (\{y | \forall x \in \mathbb{Z}.(y = x + 5)\})$, as the domains are different (and thus so is the range).

- Properties of Relations
  Here we consider a relation R on a set A.
  - **Reflexivity** R is *reflexive* if and only if $\forall x \in A.xRx$. In terms of the graph, every vertex has a self-loop.
  - **Irreflexivity** R is *irreflexive* if and only if $\forall x \in A.\neg(xRx)$. In terms of the graph, no vertex has a self-loop.
  - **Symmetry** R is *symmetric* if and only if $\forall x, y \in A.xRy \rightarrow yRx$. In terms of the graph, every edge from x to y has an edge back from y to x. Note they haven't seen this in class, so don't worry too much about it for now.
  - **Asymmetry** R is *asymmetric* if and only if $\forall x, y \in A.xRy \rightarrow \neg(yRx)$. Again, haven't seen this in class yet so don't worry too much about it.
  - **Antisymmetry** R is *antisymmetric* if and only if $\forall x, y \in A.x \neq y \wedge xRy \rightarrow \neg(yRx)$. Same as asymmetry, but with self loops.

- **Transitivity**R is *transitive* if and only if $\forall x, y, z \in A . xRy \wedge yRz \rightarrow xRz$. Students have seen how this looks on a graph in class, but might be worth refreshing that here.

- For each of the following, determine if it is 1) reflexive, 2) irreflexive, 3) transitive, 4) symmetric, 5) asymmetric
  - $R_1 = A \times A$
    1. By definition of $A \times A$, $\forall a, b \in A.(a, b) \in A \times A$. So, $(a, a) \in A \times A \forall a \in A$ which implies that aR_1a, and so R is reflexive.
    2. Since R_1 is reflexive, it is not irreflexive.
    3. Let $a, b, c \in A$. By definition of $A \times A$, $(a, b), (b, c), (a, c) \in R_1$ so $(a, b) \wedge (b, c) \rightarrow (a, c)$ is true for all elements of A so R_1 is transitive.
    4. Since $\forall a, b \in A (a, b) \in A \times A$, $(a, b) \rightarrow (b, a)$ is true. So, R_1 is symmetric.
    5. Since R_1 is symmetric, there are two elements a, b such that $aR_1b \rightarrow bR_1a$, and so R_1 is not asymmetric.
  - $f_\emptyset$ such that $f_\emptyset(a) = \emptyset \forall a \in A$
    1. Let $a \in A$. Note $(a, a) \notin f_\emptyset$, so the relation is not reflexive.
    2. Note that forall $a \in A$, $(a, a) \notin f_\emptyset$, so the relation is irreflexive.
    3. The relation is transitive since $False \rightarrow False$ is true always.
    4. By similar reasoning, the relation is symmetric.
    5. Again, since $False \rightarrow False$ is always true, the proposition $(af_) \rightarrow \neg(bf_)$ is always true, so the relation is asymmetric.
  - {(a,a),(a,b),(b,c),(c,c)} on set $A = \{a, b, c, d\}$
    1. The relation is not reflexive.
    2. The relation is not irreflexive.
    3. The relation is not transitive since $(a, b), (b, c) \in R$ but $(a, c) \notin R$.
    4. The relation is not symmetric since $(a, b) \in R$ but $(b, a) \notin R$.
    5. The relation is not asymmetric since $(a, a) \in R$ and $(a, a) \in R$
    6. Note the relation *is* antisymmetric, since antisymmetry allows self loops.
  - $<$ on set $\{1, 2, 3, 4\}$
    1. The relation is not reflexive since 1 is not < 1.
    2. The relation is irreflexive since a is not $< a$ for all a.
    3. The relation is transitive.
    4. The relation is not symmetric
    5. The relation is asymmetric
  - $\leq$ on set $\{1, 2, 3, 4\}$
    1. The relation is reflexive.
    2. Since reflexive, not irreflexive.

3. The relation is transitive.
4. The relation is not symmetric.
5. The relation is not asymmetric.
6. The relation is antisymmetric

– $\subset$ on set $2^A = \text{powerset}(A)$ for any non-empty set A

1. It is worth it to do an example. Since no set is a proper subset of itself, the operation is not reflexive.
2. Since no set is a proper subset of itself, there are no self-loops, or $\neg(aRa)$ is true for all $a \in 2^A$.
3. Subset is transitive.
4. Subset is not symmetric.
5. Subset is not asymmetric.
6. Subset is anti symmetric.

- Consider the interval in $\mathbb{R}^2$ given by $[0,3] \times [0,2]$. Give a description of the relation specified by this set.
  Answer: $R = \{(x,y)|x,y \in \mathbb{R}.0 \leq x \leq 3 \land 0 \leq y \leq 2\}$

- Let $S = A \times B$ where $A = \{1,2,3\}$ and $B = \{a,b\}$. How many possible relations on S exist?
  Answer:$S = \{(1,a),(1,b),(2,a),(2,b),(3,a),(3,b)\}$. Now, an element of a relation on S is $\{(x,y).x \in S, y \in S\} = \{((t,v),(u,w)).t,u \in A,\ v,w \in B\} = S \times S$. We want the size of the set of all possible relations of S, so the size of the powerset of $S \times S$. Note the size of $S \times S$ is 36 since there are 6 options for the first spot, and 6 for the second. The size of the powerset of $S \times S$ is 2^{36} so we have 2^{36} possible relations.

- Let $S = Q \times R$. If $|Q| = n$ and $|R| = m$, then how many possible relations are there on S?
  Answer: The number of possible relations is the number of subsets of the cross product of S. Similarly to above, there are nm elements in S. This implies there are $nmnm = (nm)^2$ elements in $S \times S$. The number of subsets of a set is the size of its powerset, which in this case is $2^{(nm)^2}$.

- Additional Exercises:

  – A is the set of all people in the world. $aRb \leftrightarrow a$ is the sister of b
  – Draw the graph of the symmetric relation on $A = \{a,b,c,d\}$, $R = \{(a,b),(b,a),(a,c),(c,a),(a,d),(d,a)\}$
  – A is the set of all triangles in the plane. t_1Rt_2 if and only if all angles of t_1 are the same as the angles of t_2.
  – A is the set of all ordered pairs of real numbers. $(a,b)R(c,d)$ if and only if $a = c$.
  – $S = \{1,2,3,4\}$ and $A = S \times S$. $(a,b)R(c,d)$ if and only if $ad = bc$