Vector Representation of Rotations

Carlo Tomasi

The vector representation of rotation introduced below is based on Euler's theorem, and has three parameters. The conversion from a rotation vector to a rotation matrix is called Rodrigues' formula, and is derived below based on geometric considerations. The inverse of Rodrigues' formula is developed as well.

1 Rotation Vectors

A rotation matrix is an array of nine numbers. These are subject to the six norm and orthogonality constraints, so only three *degrees of freedom* are left: if three of the numbers are given, the other six can be computed from these equations. In numerical optimization problems, the redundancy of rotation matrices is inconvenient, and a minimal representation of rotation is often preferable.

The simplest such representation is based on *Euler's theorem*, stating that every rotation can be described by an axis of rotation and an angle around it. A compact representation of axis and angle is a three-dimensional *rotation vector* whose direction is the axis and whose magnitude is the angle in radians. The axis is oriented so that the acute-angle rotation is counterclockwise around it. As a consequence, the angle of rotation is always nonnegative, and at most π.

While simple, the rotation-vector representation of rotation must be used with some care. As defined earlier, the set of all rotation vectors is the three-dimensional ball[1] of radius π. However, two antipodal points on the sphere, that is, two vectors $\mathbf{r}$ and $-\mathbf{r}$ with norm π, represent the same 180-degree rotation.

Whether this lack of uniqueness is a problem depends on the application. For instance, when comparing rotations, it would be troublesome if the same rotation had two different representations. To preserve uniqueness, one can carefully peel away half of the sphere from the ball, and define the *half-open rotation ball* as the following union of disjoint sets:

$$\{\mathbf{r} \,:\, \|\mathbf{r}\| < \pi\} \,\cup\, \{\mathbf{r} \,:\, \|\mathbf{r}\| = \pi \,\cap\, r_1 > 0\} \,\cup\, \{\mathbf{r} \,:\, \|\mathbf{r}\| = \pi \,\cap\, r_1 = 0 \,\cap\, r_2 > 0\} \,\cup\, \{(0,0,\pi)\} \;.$$

These sets are respectively the open unit ball, the open hemisphere with its pole at $(\pi, 0, 0)$, the open half-equator of that hemisphere centered at $(0, \pi, 0)$, and the individual point $(0, 0, \pi)$. The last three sets are illustrated in Figure 1.

The formula for finding the rotation matrix corresponding to an angle-axis vector is called *Rodrigues' formula*, which is now derived.

Let $\mathbf{r}$ be a rotation vector. If the vector is $(0, 0, 0)$, then the rotation is zero, and the corresponding matrix is the identity matrix:

$$\mathbf{r} = \mathbf{0} \rightarrow R = I \;.$$

[1] A *ball* of radius r in $\mathbb{R}^n$ is the set of points $\mathbf{p}$ such that $\|\mathbf{p}\| \leq r$. In contrast, a *sphere* of radius r in $\mathbb{R}^n$ is the set of points $\mathbf{p}$ such that $\|\mathbf{p}\| = r$.

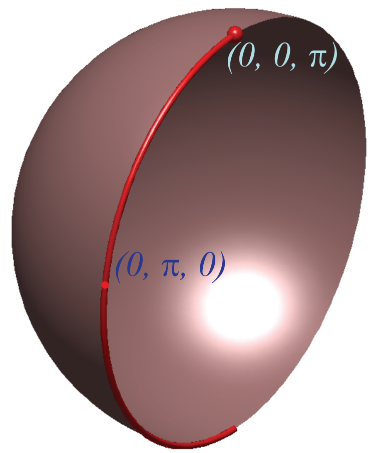

Figure 1: The parts of the sphere of radius π that are included in the half-open rotation ball. The interior of the ball is included as well, but is not shown in the figure for clarity. The pole of the hemisphere in the picture is the point $(\pi, 0, 0)$.

Let us now assume that $\mathbf{r}$ is not the zero vector. The unit vector for the axis of rotation is then

$$\mathbf{u} = \frac{\mathbf{r}}{\|\mathbf{r}\|}$$

and the angle is

$$\theta = \|\mathbf{r}\| \quad \text{radians.}$$

The rotation has no effect on a point $\mathbf{p}$ along the axis. Suppose then that $\mathbf{p}$ is off the axis. To see the effect of rotation on $\mathbf{p}$, we decompose $\mathbf{p}$ into two orthogonal vectors, one along $\mathbf{u}$ and the other perpendicular to it:

$$\mathbf{a} = P_{\mathbf{u}}\mathbf{p} = \mathbf{u}\mathbf{u}^T\mathbf{p}$$

is along $\mathbf{u}$, and

$$\mathbf{b} = \mathbf{p} - \mathbf{a} = (1 - \mathbf{u}\mathbf{u}^T)\mathbf{p}$$

is orthogonal to $\mathbf{u}$, as shown in Figure 2.

The rotation leaves $\mathbf{a}$ unaltered, and rotates $\mathbf{b}$ by θ in the plane orthogonal to $\mathbf{u}$. To express the latter rotation, we introduce a third vector

$$\mathbf{c} = \mathbf{u} \times \mathbf{p}$$

that is orthogonal to both $\mathbf{u}$ and $\mathbf{p}$, and has the same norm as $\mathbf{b}$ (because $\mathbf{u}$ is a unit vector and because of the definition of cross product). Since $\mathbf{b}$ and $\mathbf{c}$ have the same norm, the rotated version of $\mathbf{b}$ is

$$\mathbf{b}' = \mathbf{b}\cos\theta + \mathbf{c}\sin\theta \ .$$

The rotated version of the entire vector $\mathbf{p}$ is then

$$\begin{aligned}
\mathbf{p}' &= \mathbf{a} + \mathbf{b}' = \mathbf{a} + \mathbf{b}\cos\theta + \mathbf{c}\sin\theta = \mathbf{u}\mathbf{u}^T\mathbf{p} + (1 - \mathbf{u}\mathbf{u}^T)\mathbf{p}\cos\theta + \mathbf{u} \times \mathbf{p}\sin\theta \\
&= [I\cos\theta + (1 - \cos\theta)\mathbf{u}\mathbf{u}^T + \mathbf{u}_\times \sin\theta]\mathbf{p}
\end{aligned}$$

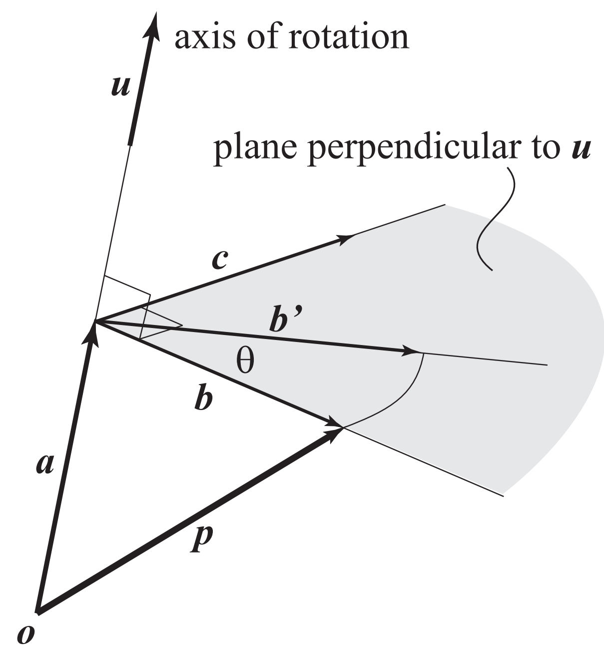

Figure 2: Vectors used in the derivation of Rodrigues' formula.

so that we have the result in the top box of Table 1. That equation is called *Rodrigues' formula*.

To invert this formula, note that the sum of its first two terms,

$$I\cos\theta + (1-\cos\theta)\mathbf{u}\mathbf{u}^T$$

is a symmetric matrix, while the last term,

$$\mathbf{u}_\times \sin\theta$$

is antisymmetric. Therefore,

$$R - R^T = 2\mathbf{u}_\times \sin\theta = \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix} \sin\theta = 2\begin{bmatrix} 0 & -\rho_3 & \rho_2 \\ \rho_3 & 0 & -\rho_1 \\ -\rho_2 & \rho_1 & 0 \end{bmatrix} .$$

Since the vector $\mathbf{u}$ has unit norm, the norm of the vector (ρ_1, ρ_2, ρ_3) is $\sin\theta$. Direct calculation shows that the *trace*, that is, the sum of the diagonal elements of the rotation matrix R, is equal to $2\cos\theta + 1$, so that

$$\cos\theta = (r_{11} + r_{22} + r_{33} - 1)/2 .$$

If $\sin\theta = 0$, and $\cos\theta = 1$ then the rotation vector is

$$\mathbf{r} = \mathbf{0} .$$

If $\sin\theta = 0$, and $\cos\theta = -1$ then Rodrigues' formula simplifies to the following:

$$R = -I + 2\mathbf{u}\mathbf{u}^T$$

so that

$$\mathbf{u}\mathbf{u}^T = \frac{R + I}{2} \ .$$

This equation shows that each of the three columns of $(R + I)/2$ is a multiple of the unknown unit vector $\mathbf{u}$. Since the norm of $\mathbf{u}$ is one, not all its entries can be zero. Let $\mathbf{v}$ be any nonzero column of $R + I$. Then

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$$

and

$$\mathbf{r} = \mathbf{u}\pi \ .$$

In the general case, $\sin\theta \neq 0$. Then, the normalized rotation vector is

$$\mathbf{u} = \frac{\rho}{\|\rho\|} \ .$$

From $\sin\theta$ and $\cos\theta$, the two-argument arc-tangent function yields the angle θ, and

$$\mathbf{r} = \mathbf{u}\theta \ .$$

The two-argument function $\arctan_2$ is defined as follows for $(x, y) \neq (0, 0)$

$$\arctan_2(y, x) = \begin{cases} \arctan(\frac{y}{x}) & \text{if } x > 0 \\ \pi + \arctan(\frac{y}{x}) & \text{if } x < 0 \\ \frac{\pi}{2} & \text{if } x = 0 \text{ and } y > 0 \\ -\frac{\pi}{2} & \text{if } x = 0 \text{ and } y < 0 \end{cases} \tag{1}$$

and is undefined for $(x, y) = (0, 0)$. This function returns the arc-tangent of y/x (notice the order of the arguments) in the proper quadrant, and extends the function by continuity along the y axis.

Table 1 summarizes this discussion.

The rotation matrix R corresponding to the rotation vector

$$\mathbf{r} \quad \text{such that} \quad \|\mathbf{r}\| \leq \pi$$

can be computed as follows:

$$\theta = \|\mathbf{r}\|$$

If $\theta = 0$, then $R = I$. Otherwise,

$$\mathbf{u} = \frac{\mathbf{r}}{\theta} \quad \text{and} \quad R = I \cos\theta + (1 - \cos\theta)\mathbf{u}\mathbf{u}^T + \mathbf{u}_\times \sin\theta \ .$$

The rotation vector $\mathbf{r}$ corresponding to the rotation matrix

$$R \quad \text{such that} \quad R^T R = R R^T = I \quad \text{and} \quad \det(R) = 1$$

can be computed as follows:

$$A = \frac{R - R^T}{2} \quad , \quad \rho = \begin{bmatrix} a_{32} & a_{13} & a_{21} \end{bmatrix}^T$$

$$s = \|\rho\| \quad , \quad c = (r_{11} + r_{22} + r_{33} - 1)/2 \ .$$

If $s = 0$ and $c = 1$, then $\mathbf{r} = \mathbf{0}$. Otherwise, if $s = 0$ and $c = -1$, let $\mathbf{v} =$ a nonzero column of $R + I$. Then,

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} \quad , \quad \mathbf{r} = S_{1/2}(\mathbf{u}\pi) \ .$$

In this expression, the function $S_{1/2}(\mathbf{r})$ flips signs of the coordinates of vector $\mathbf{r}$ (assumed here to have norm π) to force it onto the half-hemisphere of Figure 1, in order to ensure uniqueness:

$$S_{1/2}(\mathbf{r}) = \begin{cases} -\mathbf{r} & \text{if } \|\mathbf{r}\| = \pi \text{ and } ((r_1 = r_2 = 0 \text{ and } r_3 < 0) \\ & \quad \text{or } (r_1 = 0 \text{ and } r_2 < 0) \text{ or } (r_1 < 0)) \\ \mathbf{r} & \text{otherwise.} \end{cases}$$

Finally, if $\sin\theta \neq 0$,

$$\mathbf{u} = \frac{\rho}{s} \quad , \quad \theta = \arctan_2(s, c) \quad , \text{and} \quad \mathbf{r} = \mathbf{u}\theta$$

where the function $\arctan_2$ is defined in equation (1).

Table 1: Rodrigues' formula (top box) and its inverse (bottom box) transform between a rotation vector $\mathbf{r}$ and a rotation matrix R.