CPS 590.4

Computational problems, algorithms, runtime, hardness

(a ridiculously brief introduction to theoretical computer science)

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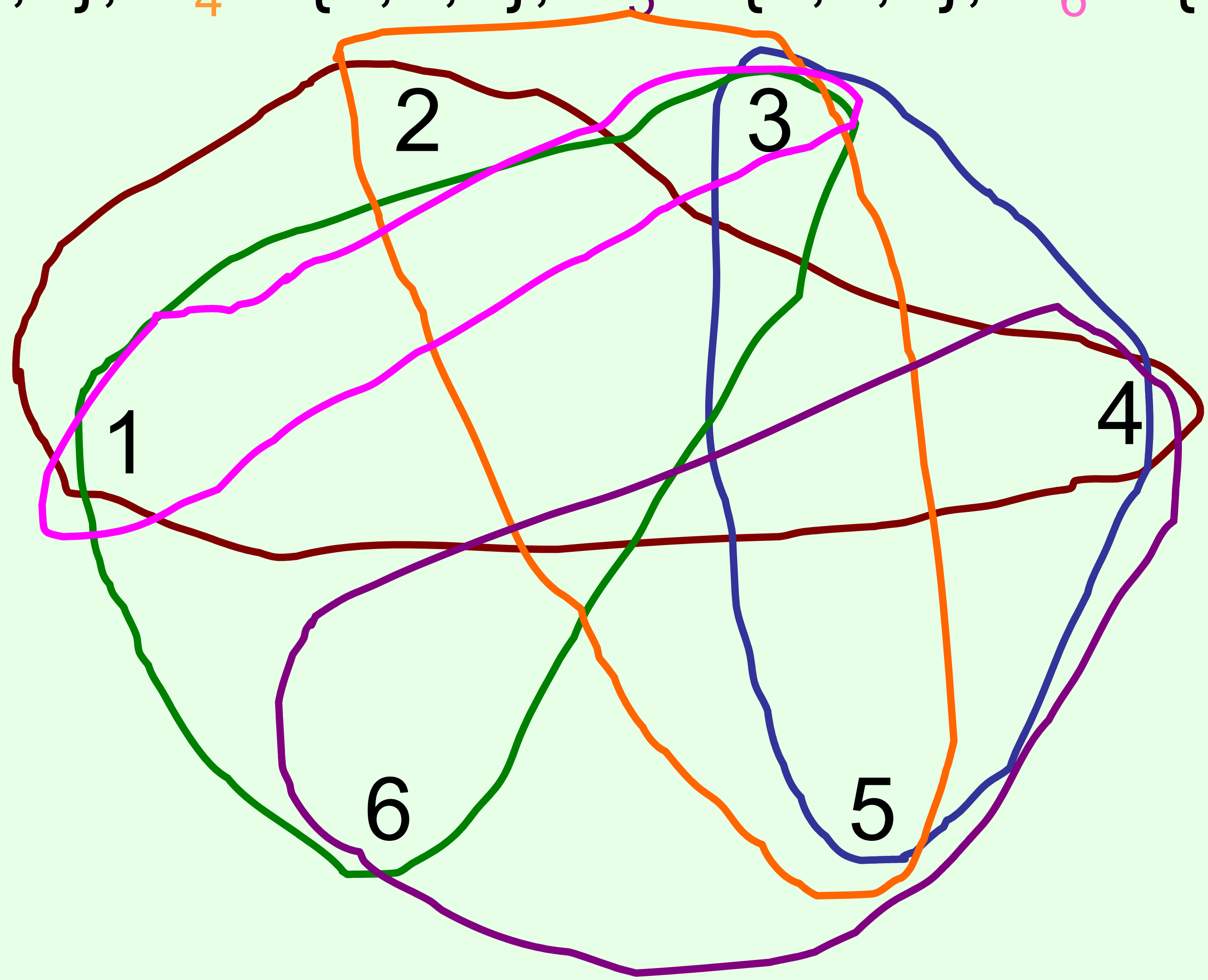
Set Cover (a computational problem)

- We are given:
  - A finite set $S = \{1, \ldots, n\}$
  - A collection of subsets of S: $S_1, S_2, \ldots, S_m$
- We are asked:
  - Find a subset T of $\{1, \ldots, m\}$ such that $\bigcup_{j \text{ in } T} S_j = S$
  - Minimize $|T|$
- Decision variant of the problem:
  - we are additionally given a target size k, and
  - asked whether a T of size at most k will suffice
- One instance of the set cover problem:

$S = \{1, \ldots, 6\}$, $S_1 = \{1,2,4\}$, $S_2 = \{3,4,5\}$, $S_3 = \{1,3,6\}$, $S_4 = \{2,3,5\}$, $S_5 = \{4,5,6\}$, $S_6 = \{1,3\}$

Visualizing Set Cover

- $S = \{1, \ldots, 6\}$, $S_1 = \{1,2,4\}$, $S_2 = \{3,4,5\}$, $S_3 = \{1,3,6\}$, $S_4 = \{2,3,5\}$, $S_5 = \{4,5,6\}$, $S_6 = \{1,3\}$



Using glpsol to solve set cover instances

- How do we model set cover as an integer program?
- See examples

Algorithms and runtime

- We saw:
  - the runtime of glpsol on set cover instances increases rapidly as the instances' sizes increase
  - if we drop the integrality constraint, can scale to larger instances
- Questions:
  - Using glpsol on our integer program formulation is but one algorithm – maybe other algorithms are faster?
    - different formulation; different optimization package (e.g., CPLEX); simply going through all the combinations one by one; …
  - What is "fast enough"?
  - Do (mixed) integer programs always take more time to solve than linear programs?
  - Do set cover instances fundamentally take a long time to solve?

A simpler problem: sorting (see associated spreadsheet)

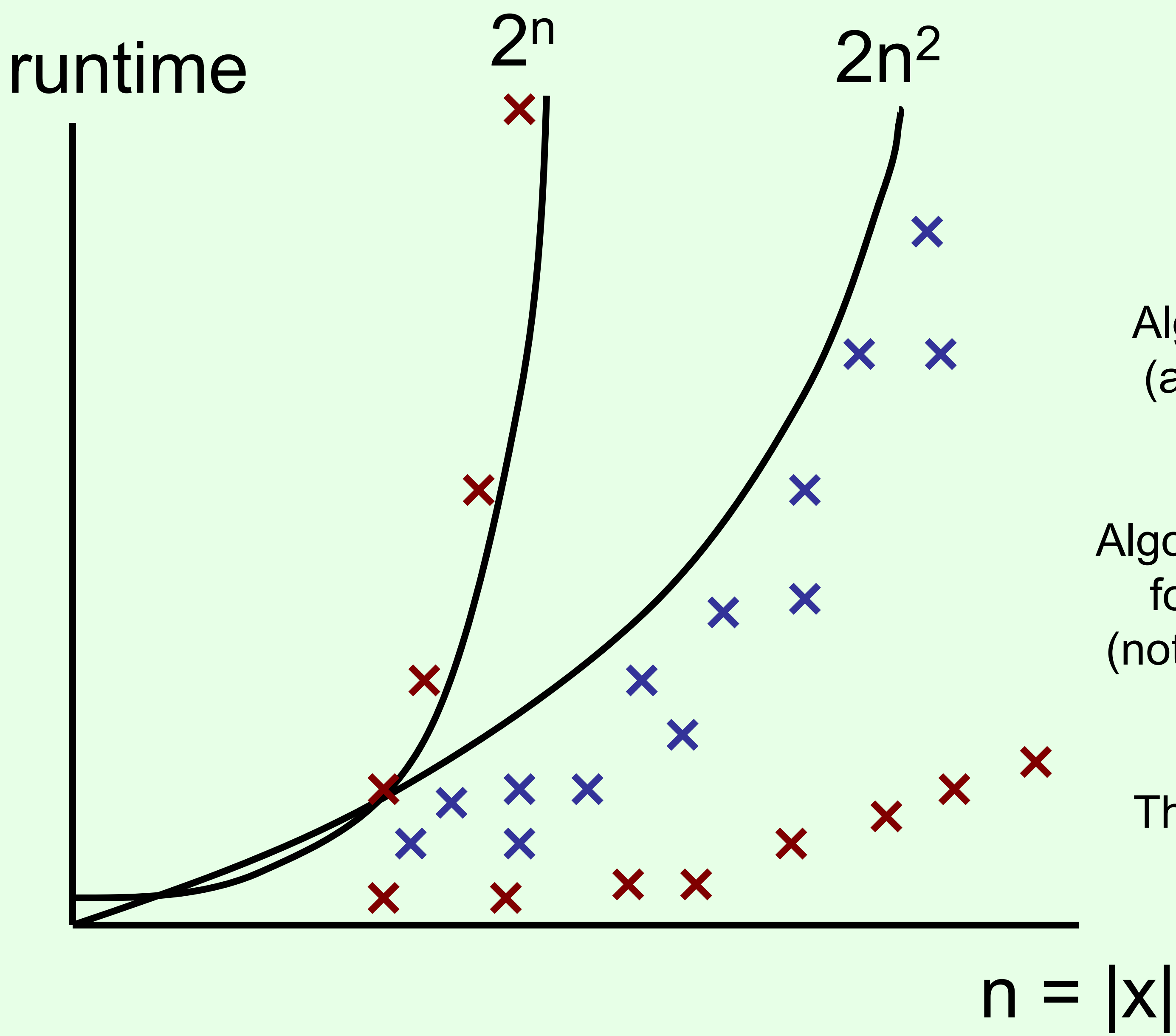
- Given a list of numbers, sort them
- **(Really) dumb algorithm:** Randomly perturb the numbers. See if they happen to be ordered. If not, randomly perturb the whole list again, etc.
- **Reasonably smart algorithm:** Find the smallest number. List it first. Continue on to the next number, etc.
- **Smart algorithm (MergeSort):**
  - It is easy to merge two lists of numbers, each of which is already sorted, into a single sorted list
  - So: divide the list into two equal parts, sort each part with some method, then merge the two sorted lists into a single sorted list
  - … actually, to sort each of the parts, we can again use MergeSort! (The algorithm “calls itself” as a subroutine. This idea is called *recursion.*) Etc.

Polynomial time

- Let |x| be the size of problem instance x (e.g., the size of the file in the .lp language)
- Let a be an algorithm for the problem
- Suppose that for any x, runtime(a,x) < cf(|x|) for some constant c and function f

  Then we say algorithm a's runtime is O(f(|x|))
- a is a polynomial-time algorithm if it is O(f(|x|)) for some polynomial function f
- P is the class of all problems that have at least one polynomial-time algorithm
- Many people consider an algorithm efficient if and only if it is polynomial-time

Two algorithms for a problem



× run of algorithm 1

× run of algorithm 2

Algorithm 1 is $O(n^2)$
(a polynomial-time algorithm)

Algorithm 2 is not $O(n^k)$ for any constant k
(not a polynomial-time algorithm)

The problem is in P

Linear programming and (mixed) integer programming

- LP and (M)IP are also computational problems
- LP is in P
  - Ironically, the most commonly used LP algorithms are not polynomial-time (but “usually” polynomial time)
- (M)IP is not known to be in P
  - Most people consider this unlikely

Reductions

- Sometimes you can reformulate problem A in terms of problem B (i.e., reduce A to B)
  - E.g., we have seen how to formulate several problems as linear programs or integer programs
- In this case problem A is at most as hard as problem B
  - Since LP is in P, all problems that we can formulate using LP are in P
  - Caveat: only true if the linear program itself can be created in polynomial time!

NP ("nondeterministic polynomial time")

- Recall: decision problems require a yes or no answer
- NP: the class of all decision problems such that if the answer is yes, there is a simple proof of that
- E.g., "does there exist a set cover of size k?"
- If yes, then just show which subsets to choose!

- Technically:
  - The proof must have polynomial length
  - The correctness of the proof must be verifiable in polynomial time

P vs. NP

- Open problem: is it true that P=NP?
- The most important open problem in theoretical computer science (maybe in mathematics?)
- $1,000,000 Clay Mathematics Institute Prize
- Most people believe P is not NP
- If P were equal to NP…
  - Current cryptographic techniques can be broken in polynomial time
  - Computers may be able to solve many difficult mathematical problems…
    - … including, maybe, some other Clay Mathematics Institute Prizes! ☺

NP-hardness

- A problem is NP-hard if the following is true:
  - Suppose that it is in P
  - Then P=NP
- So, trying to find a polynomial-time algorithm for it is like trying to prove P=NP
- Set cover is NP-hard
- Typical way to prove problem Q is NP-hard:
  - Take a known NP-hard problem Q'
  - Reduce it to your problem Q
    - (in polynomial time)
- E.g., (M)IP is NP-hard, because we have already reduced set cover to it
  - (M)IP is more general than set cover, so it can't be easier
- A problem is NP-complete if it is 1) in NP, and 2) NP-hard

Reductions:

To show problem Q is easy:

$$Q \xrightarrow{\textit{reduce}} \text{Problem known to be easy (e.g., LP)}$$

To show problem Q is (NP-)hard:

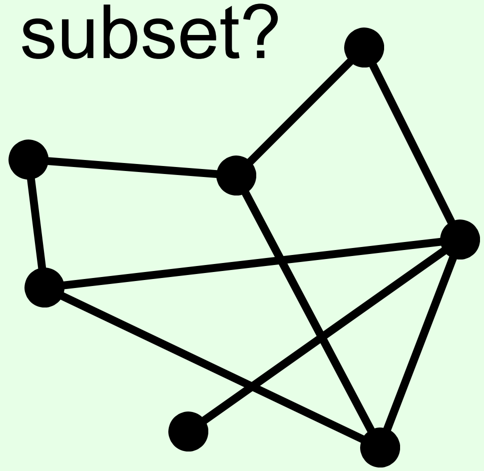
$$\text{Problem known to be (NP-)hard (e.g., set cover, (M)IP)} \xrightarrow{\textit{reduce}} Q$$

ABSOLUTELY NOT A PROOF OF NP-HARDNESS:

$$Q \xrightarrow{\textit{reduce}} \text{MIP}$$

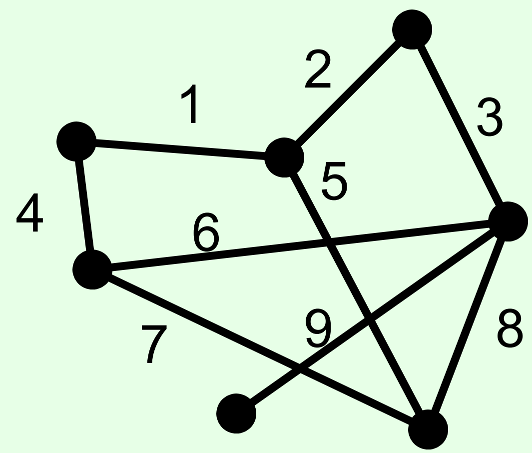
Independent Set

- In the below graph, does there exist a subset of vertices, of size 4, such that there is no edge between members of the subset?



- General problem (decision variant): given a graph and a number k, are there k vertices with no edges between them?
- NP-complete

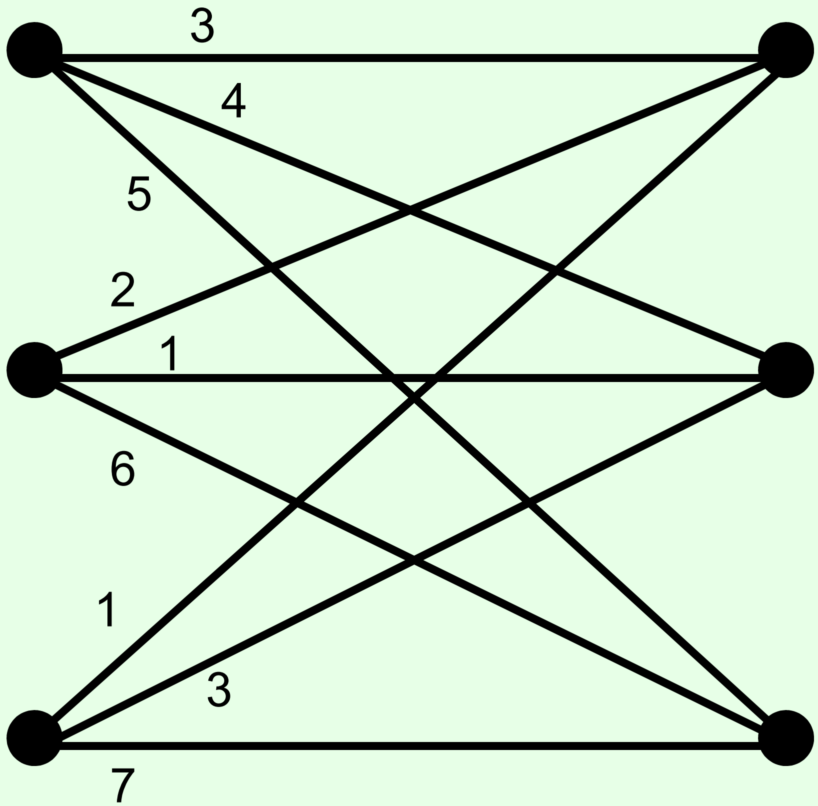
Reducing independent set to set cover



, k=4

- In set cover instance (decision variant),
  - let S = {1,2,3,4,5,6,7,8,9} (set of edges),
  - for each vertex let there be a subset with the vertex's adjacent edges: {1,4}, {1,2,5}, {2,3}, {4,6,7}, {3,6,8,9}, {9}, {5,7,8}
  - target size = #vertices - k = 7 - 4 = 3
- Claim: answer to both instances is the same (why??)
- So which of the two problems is harder?

Weighted bipartite matching



- Match each node on the left with one node on the right (can only use each node once)
- Minimize total cost (weights on the chosen edges)

Weighted bipartite matching…

- minimize $c_{ij}\, x_{ij}$
- subject to
- for every i, $\Sigma_j\, x_{ij} = 1$
- for every j, $\Sigma_i\, x_{ij} = 1$
- for every i, j, $x_{ij} \geq 0$

- Theorem [Birkhoff-von Neumann]: this linear program always has an optimal solution consisting of just integers
  - and typical LP solving algorithms will return such a solution

- So weighted bipartite matching is in P