1 Analysis

Objectives: review of background knowledge and get familiar with basic analysis tasks via SVD and EVD.

Denote by $V$ an inner-product space of dimension $n, n > 1$, over a field $F$ with inner product $\langle u, v \rangle$. Let $\| \cdot \|$ be the induced norm. Denote by $I$ the identity operator on $V$. Denote by $\circ$ operator composition. The zero $0$ denotes the zero element or the zero operator, depending on the context.

1.1 Decomposing & filtering by orthogonal projections

Let $u_1 \in V$ be a fixed/chosen vector of unit length $\|u_1\| = 1$. Then

$$P_1(v) \triangleq u_1 < u_1, v > = u_1 c_1(v) \quad P_1^\perp(v) = v - P_1(v) = (I - P_1)(v), \quad v \in V. \quad (1)$$

are the orthogonal projectors onto the subspaces span$(u_1)$ and span$^\perp(u_1)$, respectively.

1. Case: one-dimensional subspace & its complement space

(a) \[Y/N/M\]

For any $v \in V$, $v = P_1(v) + P_1^\perp(v)$ is one among many orthogonal splits of $v$ into span$(u_1)$ and span$^\perp(u_1)$.

(b) \[Y/N/M\]

The projection coefficient $c_1(v) = < u_1, v >$ is a linear functional over $V$.

(c) \[Y/N/M\]

$$(P_1 \circ P_1)(v) = P_1(v), \quad (P_1^\perp \circ P_1^\perp)(v) = P_1^\perp(v), \quad \% \text{idempotent}$$

$$(P_1 \circ P_1^\perp)(v) = (P_1^\perp \circ P_1)(v) = 0, \quad \% \text{mutual null spaces} \quad (2)$$

(d) Express $P_1(v)$ and $P_1^\perp$ in a matrix form when $V = \mathbb{R}^n$, $F = \mathbb{R}$ and $< u, v > = u^T v$.

Case: multi-dimensional subspace & its complement space

Let $\{u_j \mid j = 1, 2, \cdots, k\}$ be a set of orthonormal vectors, $1 < k < \text{dim}(V)$.

(e) \[Y/N/M\]

On sequential projections: Let $P_{u_j}(v) = u_j < u_j, v >, 1 \leq j \leq k$. Then,

$$P_{u_i} \circ P_{u_j} = P_{u_j} \circ P_{u_i} = 0, \quad P_{u_i}^\perp \circ P_{u_j}^\perp = P_{u_j}^\perp \circ P_{u_i}^\perp = 0, \quad 1 \leq i, j \leq k. \quad (3)$$

(f) \[Y/N/M\]

On parallel projections:

$$P_k(u) = U_k c_{[1:k]}(v), \quad P_k^\perp(u) = u - P_k(u), \quad (4)$$

are the orthogonal projectors to span$(U_k)$ and span$^\perp(U_k)$, respectively, where

$$U_k = [u_1, u_2, \cdots, u_k], \quad c_{[1:k]}(v) = [< u_1, v >, \cdots, < u_k, v >]^T. \quad (5)$$

The projection coefficient vector $c_{[1:k]}(v)$ is a linear operator from $V$ to $\mathbb{R}^k$.

(g) \[Y/N/M\]

For any $v \in V$, $v = P_k(v) + P_k^\perp(v)$ is the unique splot of $v$ onto span$(U_k)$ and span$^\perp(U_k)$.

(h) \[Y/N/M\]

$$(P_k \circ P_k)(v) = P_k(v), \quad (P_k^\perp \circ P_k^\perp)(v) = P_k^\perp(v), \quad \% \text{idempotent}$$

$$(P_k \circ P_k^\perp)(v) = (P_k^\perp \circ P_k)(v) = 0, \quad \% \text{mutual null spaces} \quad (6)$$

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(i) Express $P_k(v)$ in the case-specific matrix form when $V = \mathbb{R}^n$ with $< u, v > = u^T v$.

2. Elementary analysis via SVD. Let $A \in \mathbb{R}^{m \times n}$. Let $A = U_k \Sigma_k V_k^T$ be the singular value decomposition of $A$, with $\Sigma_k$ containing the nonzero singular values only, $\sigma_j \geq \sigma_{j+1}$. By scaling $A$, we may assume $\sigma_1 = 1$.

Consider the rank-deficient case $k < \min\{m, n\}$. There exists $V = [V_k, V_{n-k}]$ as a completed orthonormal basis of the input space $\mathbb{R}^n$. There exists $U = [U_k, U_{n-k}]$ as a completed orthonormal basis of $\mathbb{R}^m$. (see the next subsection for a constructive completion approach.)

(a) [Y/N/M] $\text{null}(A) = \text{span}(V_{n-k})$, $\text{null}(A^T) = \text{span}(U_{n-k})$.

(b) [Y/N/M] Let $y := Ax$. For some input $x$, a component of $x$ can not be sensed in the output $y$.

(c) [Y/N/M] Let $y_1 = Ax_1$ and $y_2 = Ax_2$. If $y_1 \neq y_2$, then $x_1 \neq x_2$; and vice versa.

1.2 Householder reflection: geometry, properties & impacts

Let $V, < \cdot, \cdot >$ and $U_k$ be as described in the previous subsection.

$$H_k(v) \triangleq P_k^+ (v) - P_k (v) = (I - 2 P_k) (v)$$

In the reflection, $P_k^+ (v)$ of vector $v$ is preserved and $P_k (v)$ is flipped to the opposite direction. The reflection is involuntary, i.e., $H_k \circ H_k = I$.

3. Constructive orthogonalization & completion of a subspace basis.

In the space $\mathbb{V} = \mathbb{R}^m$, denote a set of $k$ linearly independent vectors by $A \in \mathbb{R}^{m \times k}$, $k < m$.

Recall that there exist a sequence of Householder transforms and/or permutations so that

$$H(u_k) \cdots H(u_2)H(u_1)A = \begin{bmatrix} R_k \\ 0_{(m-k) \times k} \end{bmatrix}$$

where $R_k$ is upper triangular and nonsingular. Let $Q = H(u_1)H(u_2) \cdots H(u_k)$.

(a) [Y/N/M] $A = Q(:,1:k)R_k$ is the QR factorization of $A$;

$Q(:,1:k)$ is an orthonormal basis of span($A$).

(b) [Y/N/M] $Q(:,k+1:m)$ is an orthonormal basis of span$(A)$.

Remarks: The constructive completion of an orthogonal basis by the Householder transform is a major analytical advantage over the Gram-Schmidt process. Another major and more important advantage is in numerical stability.

(c) Assume the SVD of $R_k$ is available, $R_k = V_k \Sigma_k V_k^T$.

Describe the SVD of $A$ via its QR factorization.

(d) Assume the SVD of $A$ is available, $A = U_k \Sigma_k V_k^T$.

Describe the EVDs of $A^T A$, $A A^T$, $\exp(-A^T A)$ and $\exp(-AA^T)$, respectively.
1.3 Sensitivity & perturbation analysis via SVD

Assume that there exists solution to the linear system of equations $Ax = b$ (such as the system resulted from a linear LS problem). Denote by $\hat{x}$ a computed solution. Define its residual as follows,

$$r(\hat{x}) = b - A\hat{x}, \quad A \in \mathbb{R}^{m \times n}. \quad (9)$$

The residual is computationally available. Let $A = U_k\Sigma_kV_k^T$ be the SVD of $A$, with $\Sigma_k$ containing nonzero singular values only. Assume by scaling that $\sigma_1 = 1$.

4. \[Y/N/M\]

By the assumed solution existence, $b = U_kc_b$, where $c_b$ is unique.

Any solution with zero residual can be analytically expressed as follows,

$$x = V_k\Sigma_k^{-1}c_b + V_{n-k}c_d, \quad \|x\|^2 = \|\Sigma_k^{-1}c_b\|^2 + \|c_d\|^2, \quad (10)$$

for some vector $c_d$, where $V_{n-k}$ completes $V_k$ in $\mathbb{R}^n$, span$(V_{n-k}) = \text{null}(A)$.

The ideal solution has zero component in $\text{null}(A)$,

$$x^* = V_k\Sigma_k^{-1}c_b, \quad (11)$$

i.e., it is the unique solution with the minimal length ($c_d = 0$).

5. Analytical expressions for $\hat{x}$, $r(\hat{x})$ and $\epsilon(\hat{x}) = \hat{x} - x^*$, and their relationships.

(a) \[Y/N/M\]

The computed solution $\hat{x}$:

$$\hat{x} = V_k\Sigma_k^{-1}c_1 + V_{n-k}c_2, \quad \text{for some } c_1 \text{ and } c_2. \quad (12)$$

(b) \[Y/N/M\]

The residual:

$$r(\hat{x}) = U_k(c_b - c_1), \quad \|r(\hat{x})\|^2 = \|c_b - c_1\|^2. \quad (13)$$

(c) \[Y/N/M\]

The difference from the ideal solution:

$$\epsilon(\hat{x}) = \hat{x} - x^* = V_k\Sigma_k^{-1}(c_1 - c_b) + V_{n-k}c_2, \quad \|\epsilon(\hat{x})\|^2 = \|\Sigma_k^{-1}(c_b - c_1)\|^2 + \|c_2\|^2. \quad (14)$$

(d) \[Y/N/M\]

When $A$ is of full column rank, $k = n$,

$$1 \leq \frac{\|\epsilon(\hat{x})\|}{\|r(\hat{x})\|} \leq \kappa(A) = \frac{\sigma_1}{\sigma_k} \quad (15)$$

where $\kappa(A)$ is the condition number of $A$.

Remark: a small residual does not necessarily indicate an accurate solution.

(e) Assume the condition number $\kappa(A)$ is big, say, $10^6$ or bigger.

Give a worst-case scenario with $\hat{x}$ such that the ratio $\|\epsilon\|/\|r\|$ reaches the upper bound in (15).

Give a lucky-case scenario where the ratio is close to the lower bound.

(f) Optional. Extend the worst-case analysis to the situation $A$ is rank-deficient.
6. [Y/N/M] 
For Perturbation analysis. Let \( \epsilon > 0 \) be arbitrarily small. 
There exists perturbation \( E \) bounded by \( \epsilon \), \( ||E||_F < \epsilon \), such that \( \text{rank}(A + E) = \min\{m, n\} \).

Reversely, if \( A \) is not rank-deficient but with \( \sigma_k < \epsilon \). Then, there exists perturbation \( E \) bounded by \( \epsilon \), \( ||E||_F < \epsilon \), such that \( \text{rank}(A + E) < \min\{m, n\} \).

7. [Y/N/M] 
For approximation by truncation: by the SVD \( A = \sum_{j=1: k} \sigma_j u_j v_j^T \),
\[
||\sigma_j u_j v_j^T||_F = \sigma_j, \quad ||A||_F^2 = \sum_{1 \leq j \leq k} \sigma_j^2, \quad \% \text{ energy terms and total}
\]
\[
||A||_2 = \sigma_1, \quad \% \text{ spectral norm}
\]
(16)

1.4 Fixed point iterations: algorithm & analysis

Consider computational solution to the system of equations
\[
F(x) = 0, \quad F : \mathbb{R}^n \rightarrow \mathbb{R}^n, \quad n \geq 1.
\]
(17)

where \( F \) is linear or non-linear in \( x \). A solution is an equilibrium point (or root). Such a system emerges frequently in optimization problems, among other applications. For any other vector \( x_s \), the residual \( F(x_s) \neq 0 \) indicates the departure of \( x_s \) from any equilibrium point. With an iterative solution method, we generate a sequence \( \{x_k\} \), starting from an initial guess \( x_0 \),
\[
x_{k+1} = x_k + B(x_k)^{-1}F(x_k),
\]
(18)
so that the residual sequence \( \{F(x_k)\} \) converges to zero and the sequence \( \{x_k\} \) converges to an equilibrium point. Here, \( B \) is a mechanism designed to ensure the convergence, it transforms the residual \( F(x_k) \) (as the feedback) to the update in the current iterate, \( \Delta x_k = x_{k+1} - x_k \).

8. (a) [Y/N/M] 
For \( \{x_k\} \) to converge to some point (not necessarily to a root of \( F \)), it is necessary but insufficient that \( x_{k+1} - x_k \rightarrow 0 \) as \( k \rightarrow \infty \).\(^1\)

(b) [Y/N/M] 
If the iteration (18) converges with certain initial guess \( x_o \), then the following prediction-correction iteration converges with the same initial guess \( x_0 \),
\[
x_{k+1} = x_K + \Delta_p(x_k) + \Delta_c(x_k),
\]
\[
\Delta_p(x_k) = B(x_k)^{-1}F(x_k),
\]
\[
\Delta_c(x_k) = B(x_k + \Delta_p(x_k))^{-1}F(x_k + \Delta_p(x_k))^{-1}
\]
(19)

(c) Provide at least three termination criteria, without the knowledge of any true solution.

9. [Y/N/M] 
The basin of attraction with the Newton-Raphson iteration for root extraction:
Let \( k > 1 \) be an integer. \( f(x) = x^k - \alpha = 0, \alpha \in \mathbb{C} \setminus \{0\} \),

Assume \( x_o \) is in a basin of attraction. If \( x_o' \) is in a small enough neighborhood of \( x_o \), then \( x_o' \) belongs to the same basin of attraction.

\(^1\)It is a key termination condition \( ||x_{k+1} - x_k|| < \tau \), for some tolerance \( \tau \), for any iterative method.
10. Characteristic analysis of a linear $n$-term recurrence sequence, $n > 0,$

$$y(k + n) = -c^T y(k + [n]), \quad k = 0, 1, 2, \cdots$$

$$c^T = [c_0, c_1, \cdots, c_{n-1}], \quad [n] \triangleq [0:n-1],$$

with the initial segment $y([n])$ provided.

(a) [Y/N/M]

The consecutive $n$-length segments can be related via the companion matrix $C,$

$$y((k + 1) + [n]) = C y(k + [n]), \quad C = [0, I_n(:, 1:n-1)] - e_n c^T.$$  \hfill (21)

Any $n$-length segment of $y$ can be traced back to the initial segment,

$$y(k + [n]) = C^k y([n]),$$  \hfill (22)

Note: The characteristic polynomial of the recurrence is

$$p(x) = \det(xI - C) = x^n + \sum_{j=0}^{n-1} c_j x^j.$$

(b) [Y/N/M]

For any eigen-pair of $C,$ $Cv_j = \lambda_j v_j,$ $x_j$ is the Vandermonde vector at $\lambda_j,$

$$v_j = [1, \lambda_j, \lambda_j^2, \cdots, \lambda_j^{n-1}]^T.$$  \hfill (23)

Consequently, if the eigenvalues (roots of $p(x)$) are distinct, the eigenvector matrix $V = [v_1, \cdots, v_n]$ is the (transposed) Vandermonde matrix at the eigenvalues, and

$$y(k + [n]) = (V \Lambda^k V^{-1}) y([n]), \quad \Lambda = \text{diag}(\lambda_j, j = 1 : n).$$  \hfill (24)

(c) [Y/N/M]

Depending on the recurrence coefficients and the initial segment $y([n])$, a recurrence sequence may blow up (quickly or slowly), or diminish (quickly or slowly), or oscillate, as $k$ increases.

(d) Optional.

Verify first that if $p(x) = x^n - 1,$ the companion matrix is the (cyclic) shifting matrix $S,$ and the eigenvector matrix is the DFT matrix $F_n,$ $S = F_n \Lambda_n \bar{F}_n/n,$ $F$ denotes the conjugate of $F.$

Verify next that any circulant matrix is a polynomial of $S,$ $p(S) = F_n p(\Lambda_n) \bar{F}_n/n.$ This is the base for the Discrete Convolution Theorem for all discrete circulant convolutions.