

**Assignment 8**

Due Date: November 20, 11:59pm

**Problem 1:** [10pts] Given a ground set $U = \{e_1, e_2, \ldots, e_n\}$, a collection of sets $S = \{S_1, S_2, \ldots, S_m\}$ where each $S_j \subseteq U$ and $\bigcup_{j=1}^{m} S_j = U$, and a weight function $w : S_j \rightarrow \mathbb{R}^+$, we want to find a set cover $C \subseteq S$ such that $\bigcup_{S_j \in C} S_j = U$ and the total weight of sets in $C$ is minimized. Below is a linear program relaxation for the problem.

\[
\begin{align*}
\min & \quad \sum_{j=1}^{m} w(S_j) \cdot x_j \\
\text{subject to} & \quad \sum_{e_i \in S_j} x_j \geq 1 \quad \text{for } 1 \leq i \leq n \\
& \quad x_j \geq 0 \quad \text{for } 1 \leq j \leq m.
\end{align*}
\]

Let $f$ be the maximum number of sets in $S$ in which an element of $U$ appears. Consider the algorithm for this problem that first computes an optimal fractional solution $x^*$ then returns $\{S_j \mid x_j^* \geq 1/f\}$. Prove this algorithm returns a feasible solution and prove the best approximation guarantee for it that you can.

**Problem 2:** [10pts] Let $G$ be an undirected cycle with $n$ vertices. Write the Laplacian matrix of $G$. Prove that its eigenvalues and eigenvectors are

\[
\lambda_k = 2 - 2 \cos \left( \frac{k \pi}{n} \right) \quad \text{and} \quad u_k = \begin{bmatrix}
\cos \left( \frac{0 \pi}{n} \right) \\
\cos \left( \frac{2 \pi}{n} \right) \\
\cos \left( \frac{4 \pi}{n} \right) \\
\vdots \\
\cos \left( \frac{2(n-1) \pi}{n} \right)
\end{bmatrix}
\]

respectively, for $0 \leq k \leq n - 1$.

**Problem 3:** [10pts] Let $U$ be the universe of items to be hashed, and $m$ be the size of the hash table. A family $\mathcal{H}$ of hash functions is uniform if choosing a hash function uniformly at random from $\mathcal{H}$ makes every hash value equally likely for every item in the universe:

\[
\Pr_{h \in \mathcal{H}}[h(x) = i] = \frac{1}{m} \quad \text{for all } x \in U \text{ and all } i \in [m],
\]

and near-uniform if the probability is bounded by at most $2/m$. A family $\mathcal{H}$ of hash functions is universal if, for any two items in the universe, the probability of collision is as small as possible, i.e.

\[
\Pr\{h(x) = h(y) \mid x \neq y\} \leq \frac{1}{m},
\]

and near-universal if the probability of collision is at most $2/m$.

(a) [5pts] Describe a set of hash functions that is uniform but not (near-)universal.

(b) [5pts] Describe a set of hash functions that is universal but not (near-)uniform.