Game Theory

Instructor: Vincent Conitzer
Penalty kick example

Is this a "rational" outcome? If not, what is?
Real-world security applications

Airport security
Where should checkpoints, canine units, etc. be deployed?

Federal Air Marshals
Which flights get a FAM?

US Coast Guard
Which patrol routes should be followed?

Wildlife Protection
Where to patrol to catch poachers or find their snares?

Milind Tambe’s TEAMCORE group (USC→Harvard)
**Rock-paper-scissors**

- **Row player** aka. player 1 chooses a row
- **Column player** aka. player 2 (simultaneously) chooses a column

<table>
<thead>
<tr>
<th></th>
<th>0, 0</th>
<th>-1, 1</th>
<th>1, -1</th>
</tr>
</thead>
<tbody>
<tr>
<td>1, -1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-1, 1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1, -1</td>
<td></td>
<td></td>
<td>0, 0</td>
</tr>
</tbody>
</table>

A row or column is called an action or (pure) strategy.

Row player’s utility is always listed first, column player’s second.

**Zero-sum** game: the utilities in each entry sum to 0 (or a constant)
Three-player game would be a 3D table with 3 utilities per entry, etc.
A poker-like game

1 gets King  
player 1  
raise  
player 1  
nature  
player 2  
check  
raise  
check  
player 2  
call  
fold  
call  
fold  
fold  
fold  
2  
1  
1  
1  
-2  
1  
-1  
1

- cc  
- cf  
- fc  
- ff

rr  
0, 0  
0, 0  
1, -1  
1, -1

rc  
.5, -.5  
1.5, -1.5  
0, 0  
1, -1

cr  
-.5, .5  
-.5, .5  
1, -1  
1, -1

cc  
0, 0  
1, -1  
0, 0  
1, -1
“Chicken”

- Two players drive cars towards each other
- If one player goes straight, that player wins
- If both go straight, they both die

<table>
<thead>
<tr>
<th></th>
<th>D</th>
<th>S</th>
</tr>
</thead>
<tbody>
<tr>
<td>D</td>
<td>0,0</td>
<td>-1,1</td>
</tr>
<tr>
<td>S</td>
<td>1,-1</td>
<td>-5,-5</td>
</tr>
</tbody>
</table>

not zero-sum
“2/3 of the average” game

• Everyone writes down a number between 0 and 100
• Person closest to 2/3 of the average wins
• Example:
  – A says 50
  – B says 10
  – C says 90
  – Average(50, 10, 90) = 50
  – 2/3 of average = 33.33
  – A is closest (|50-33.33| = 16.67), so A wins
Rock-paper-scissors – Seinfeld variant

MICKEY: All right, rock beats paper! (Mickey smacks Kramer's hand for losing)
KRAMER: I thought paper covered rock.
MICKEY: Nah, rock flies right through paper.
KRAMER: What beats rock?
MICKEY: (looks at hand) Nothing beats rock.

<table>
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<th>1, -1</th>
<th>1, -1</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1, 1</td>
<td>0, 0</td>
<td>-1, 1</td>
<td>1, -1</td>
</tr>
<tr>
<td>-1, 1</td>
<td>1, -1</td>
<td>0, 0</td>
<td>0, 0</td>
</tr>
</tbody>
</table>
Dominance

• Player i’s strategy $s_i$ strictly dominates $s_i'$ if
  – for any $s_{-i}$, $u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$

• $s_i$ weakly dominates $s_i'$ if
  – for any $s_{-i}$, $u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$; and
  – for some $s_{-i}$, $u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$

$i = “the player(s) other than i”$
**Prisoner’s Dilemma**

- Pair of criminals has been caught
- District attorney has evidence to convict them of a minor crime (1 year in jail); knows that they committed a major crime together (3 years in jail) but cannot prove it

**Offers them a deal:**
- If both confess to the major crime, they each get a 1 year reduction
- If only one confesses, that one gets 3 years reduction

<table>
<thead>
<tr>
<th></th>
<th>confess</th>
<th>don’t confess</th>
</tr>
</thead>
<tbody>
<tr>
<td>confess</td>
<td>-2, -2</td>
<td>0, -3</td>
</tr>
<tr>
<td>don’t confess</td>
<td>-3, 0</td>
<td>-1, -1</td>
</tr>
</tbody>
</table>
"Should I buy an SUV?"

- Purchasing + gas cost
  - SUV: 5
  - Sedan: 3

- Accident cost
  - SUV: 5
  - Sedan: 2

<table>
<thead>
<tr>
<th></th>
<th>SUV</th>
<th>Sedan</th>
</tr>
</thead>
<tbody>
<tr>
<td>-10</td>
<td>-10</td>
<td>-7</td>
</tr>
<tr>
<td>-11</td>
<td>-7</td>
<td>-8</td>
</tr>
</tbody>
</table>
Back to the poker-like game

1 gets King
player 1
raise
player 2
call
fold
1 gets Jack
player 1
check
raise
player 2
call
fold

"nature"

<table>
<thead>
<tr>
<th></th>
<th>cc</th>
<th>cf</th>
<th>fc</th>
<th>ff</th>
</tr>
</thead>
<tbody>
<tr>
<td>rr</td>
<td>0, 0</td>
<td>0, 0</td>
<td>1, -1</td>
<td>1, -1</td>
</tr>
<tr>
<td>rc</td>
<td>.5, -.5</td>
<td>1.5, -1.5</td>
<td>0, 0</td>
<td>1, -1</td>
</tr>
<tr>
<td>cr</td>
<td>-.5, .5</td>
<td>-.5, .5</td>
<td>1, -1</td>
<td>1, -1</td>
</tr>
<tr>
<td>cc</td>
<td>0, 0</td>
<td>1, -1</td>
<td>0, 0</td>
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</table>
## Iterated dominance

- **Iterated dominance**: remove (strictly/weakly) dominated strategy, repeat
- **Iterated strict dominance** on Seinfeld’s RPS:

<table>
<thead>
<tr>
<th></th>
<th>0, 0</th>
<th>1, -1</th>
<th>1, -1</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1, 1</td>
<td>0, 0</td>
<td>-1, 1</td>
<td>0, 0</td>
</tr>
</tbody>
</table>

\[
\begin{array}{ccc}
0, 0 & 1, -1 & 1, -1 \\
-1, 1 & 0, 0 & -1, 1 \\
-1, 1 & 1, -1 & 0, 0 \\
\end{array}
\]

\[
\begin{array}{cc}
0, 0 & 1, -1 \\
-1, 1 & 0, 0 \\
\end{array}
\]
“2/3 of the average” game revisited

\[
\begin{align*}
& (2/3) \times 100 \\
& (2/3) \times (2/3) \times 100 \\
& \vdots \\
& 0
\end{align*}
\]

\{ dominated \}

\{ dominated after removal of (originally) dominated strategies \}
Mixed strategies

- **Mixed strategy** for player i = probability distribution over player i’s (pure) strategies
- E.g. $\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$
- Example of dominance by a mixed strategy:

<table>
<thead>
<tr>
<th></th>
<th>3, 0</th>
<th>0, 0</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0, 0</td>
<td>3, 0</td>
<td></td>
</tr>
<tr>
<td>1/2</td>
<td>1, 0</td>
<td>1, 0</td>
</tr>
</tbody>
</table>
• A profile (= strategy for each player) so that no player wants to deviate

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</tr>
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</tr>
<tr>
<td>S</td>
<td>1, -1</td>
<td>-5, -5</td>
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</tbody>
</table>

• This game has another Nash equilibrium in mixed strategies…
Rock-paper-scissors

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<tbody>
<tr>
<td>1, -1</td>
<td>0, 0</td>
<td>-1, 1</td>
<td></td>
</tr>
<tr>
<td>-1, 1</td>
<td>1, -1</td>
<td>0, 0</td>
<td></td>
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- Any pure-strategy Nash equilibria?
- But it has a mixed-strategy Nash equilibrium:
  Both players put probability 1/3 on each action
- If the other player does this, every action will give you expected utility 0
  - Might as well randomize
Nash equilibria of “chicken”...

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- Is there a Nash equilibrium that uses mixed strategies? Say, where player 1 uses a mixed strategy?
- If a mixed strategy is a best response, then all of the pure strategies that it randomizes over must also be best responses.
- So we need to make player 1 indifferent between D and S.
- Player 1’s utility for playing D = \(-p^c_S\)
- Player 1’s utility for playing S = \(p^c_D - 5p^c_S = 1 - 6p^c_S\)
- So we need \(-p^c_S = 1 - 6p^c_S\) which means \(p^c_S = \frac{1}{5}\)
- Then, player 2 needs to be indifferent as well.
- Mixed-strategy Nash equilibrium: ((4/5 D, 1/5 S), (4/5 D, 1/5 S))
  - People may die! Expected utility -1/5 for each player
The presentation game

Put effort into presentation (E)

Do not put effort into presentation (NE)

<table>
<thead>
<tr>
<th>Pay attention (A)</th>
<th>Do not pay attention (NA)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2, 2</td>
<td>-1, 0</td>
</tr>
<tr>
<td>-7, -8</td>
<td>0, 0</td>
</tr>
</tbody>
</table>

- Pure-strategy Nash equilibria: (E, A), (NE, NA)
- Mixed-strategy Nash equilibrium:
  
  ((4/5 E, 1/5 NE), (1/10 A, 9/10 NA))
  - Utility -7/10 for presenter, 0 for audience
Back to the poker-like game, again

- To make player 1 indifferent between $rr$ and $rc$, we need:
  utility for $rr = 0 \cdot P(cc) + 1 \cdot (1 - P(cc)) = 0.5 \cdot P(cc) + 0 \cdot (1 - P(cc)) = $ utility for $rc$
  That is, $P(cc) = 2/3$
- To make player 2 indifferent between $cc$ and $fc$, we need:
  utility for $cc = 0 \cdot P(rr) + (-0.5) \cdot (1 - P(rr)) = -1 \cdot P(rr) + 0 \cdot (1 - P(rr)) = $ utility for $fc$
  That is, $P(rr) = 1/3$
Commitment

• Consider the following (normal-form) game:

<table>
<thead>
<tr>
<th></th>
<th>2, 1</th>
<th>4, 0</th>
</tr>
</thead>
<tbody>
<tr>
<td>1, 0</td>
<td>3, 1</td>
<td></td>
</tr>
</tbody>
</table>

• How should this game be played?

• Now suppose the game is played as follows:
  – Player 1 commits to playing one of the rows,
  – Player 2 observes the commitment and then chooses a column

• What is the optimal strategy for player 1?

• What if 1 can commit to a mixed strategy?
Commitment as an extensive-form game

- For the case of committing to a pure strategy:

```
Player 1
   Up
   |   Down
   |   
Player 2
   Left
   |   Right
   |   
Player 2
   Left
   |   Right
   |   
2, 1
4, 0
1, 0
3, 1
```
Commitment as an extensive-form game

- For the case of committing to a mixed strategy:

  $\begin{pmatrix}
  1 & 0 \\
  .5 & .5
  \end{pmatrix}$

- Infinite-size game; computationally impractical to reason with the extensive form here
Solving for the optimal mixed strategy to commit to

[Conitzer & Sandholm 2006, von Stengel & Zamir 2010]

• For every column c separately, we will solve separately for the best mixed row strategy (defined by $p_r$) that induces player 2 to play c

• maximize $\sum_r p_r u_1(r, c)$

• subject to
  
  for any $c'$, $\sum_r p_r u_2(r, c) \geq \sum_r p_r u_2(r, c')$

  $\sum_r p_r = 1$

• (May be infeasible, e.g., if c is strictly dominated)

• Pick the c that is best for player 1
Visualization

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>C</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>U</td>
<td>0,1</td>
<td>1,0</td>
<td>0,0</td>
</tr>
<tr>
<td>M</td>
<td>4,0</td>
<td>0,1</td>
<td>0,0</td>
</tr>
<tr>
<td>D</td>
<td>0,0</td>
<td>1,0</td>
<td>1,1</td>
</tr>
</tbody>
</table>

(0,1,0) = M
(1,0,0) = U
(0,0,1) = D