## Exercises

The credit assignment reflects a subjective assessment of difficulty. A typical question can be answered using knowledge of the material combined with some thought and analysis.

1. **Deciding connectivity** (one credit). Given a simple graph with $n$ vertices and $m$ edges, the disjoint set system takes $O((n+m)\alpha(n))$ time to decide whether or not the graph is connected.

   (i) Give an algorithm that makes the same decision in time $O(n+m)$.
   (ii) Modify the algorithm so it computes the connected components in time $O(n+m)$.

2. **Shelling disks** (three credits). Consider a triangulation of a simple closed polygon in the plane, but one that may have vertices in the interior of the disk. A *shelling* is a total order of the triangles such that the union of the triangles in any initial sequence is a closed disk. Prove that every such triangulation has a shelling.

3. **Jordan curve** (two credits). Recall the Jordan Curve Theorem which says that every simple closed curve in the plane decomposes $\mathbb{R}^2$ into two.

   (i) Show the same is true for a simple closed curve on the sphere, $\mathbb{S}^2 = \{x \in \mathbb{R}^2 \mid \|x\| = 1\}$.
   (ii) Give an example that shows the result does not hold for simple closed curves on the torus.

4. **Homeomorphisms** (two credits). Give explicit homeomorphisms to show that the following spaces with topologies inherited from the respective containing Euclidean spaces are homeomorphic:

   - $\mathbb{R}^1 = \mathbb{R}$ , the real line;
   - $(0,1)$, the open interval;
   - $\mathbb{S}^1 - \{(0,1)\}$, the circle with one point removed.

   Generalize your homeomorphisms to show the same for the Euclidean plane, the open disk, and the sphere with one point removed.

5. **Splitting a link** (two credits). Prove that the Borromean rings are not splittable.

6. **Deforming a link** (two credits). Use Reidemeister moves to demonstrate that the two links in Figure I.13 are equivalent.

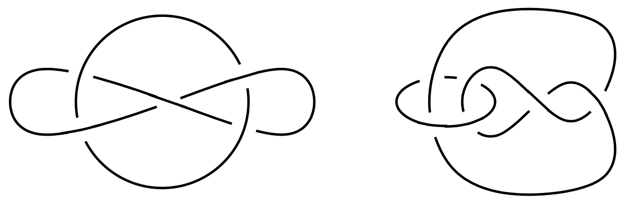

Figure I.13: Two generic projections of the Whitehead link.

7. **Planar graph coloring** (two credits). Recall that every planar graph has a vertex of degree at most five. We can use this fact to show that every planar graph has a vertex 6-coloring, that is, a coloring of each vertex with one of six colors such that any two adjacent vertices have different colors. Indeed, after removing a vertex with fewer than six neighbors we use induction to 6-color the remaining graph and when we put the vertex back we choose a color that differs from the colors of its neighbors. Refine the argument to prove that every planar graph has a vertex 5-coloring.

8. **Edge coloring** (three credits). Consider a planar graph such that all regions in the embedding in the plane are bounded by exactly three edges, including the outer region. We color each edge with one of three colors such that each region has all three different colors in its boundary.

   (i) Show that a 4-coloring of the vertices implies a 3-coloring of the edges.

   (ii) Show that a 3-coloring of the edges implies a 4-coloring of the vertices.

   In other words, proving that every planar graph has a vertex 4-coloring is equivalent to proving that every triangulation in the plane has an edge 3-coloring.