VII.1 Complexity of Reidemeister Moves

Recall the definition of Reidemeister moves from Chapter I. As proved by Reidemeister in the first half of the twentieth century, any generic projection of a knot can be transformed into any other generic projection of the same knot by a sequence of such moves [1]. In particular, for any generic projection of the unknot there is a sequence of Reidemeister moves that eliminates all crossings. This suggests a graph search algorithm to decide whether or not two generic projections describe the same knot. The only difficulty with this approach is that we do not know how long such a sequence of moves may get. We also do not know how many crossings we can expect for intermediate projections. For example, the know in Figure VII.1 is the unknot but to get it into a crossing-free projection we need to first increase the number of crossings beyond the seven in the drawing.

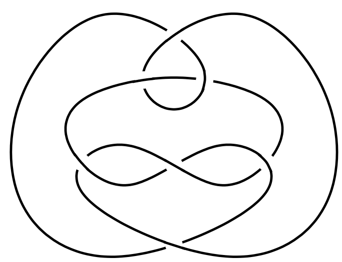

Figure VII.1: A generic projection of the unknot.

Given generic projections P and Q of the same knot, let $x(P,Q)$ be the minimum, over all Reidemeister moves transforming P to Q, of the maximum number of crossings of any projection in the sequence. Let now $x(n)$ be the maximum $x(P,Q)$, over all pairs P and Q in which P and Q have at most n crossings each. In other words, we can transform P into Q while staying below $x(n)+1$ crossings at all times.

Open question. Is there a positive constant c such that $x(n) \leq n + c$ for all c? Or less ambitiously, is $x(n)$ bounded from above by a polynomial in n?

It would be rather surprising if the answer to the first question were in the affirmative but perhaps it is to the second question.

[1] K. Reidemeister. Knotentheorie. In *Ergebnisse der Mathematik und ihrer Grenzgebiete*, Springer, Berlin, Germany, 1932.