I.2 Curves and Knots

Topology gets quickly more complicated when the dimension increases. The first step up from (0-dimensional) points are (1-dimensional) curves. Classifying them intrinsically is straightforward but can be complicated extrinsically.

Curve models. A *homeomorphism* between two topological spaces is a continuous bijection $h : \mathbb{X} \to \mathbb{Y}$ whose inverse is also continuous. If such a map exists then $\mathbb{X}$ and $\mathbb{Y}$ are considered the same. More formally, $\mathbb{X}$ and $\mathbb{Y}$ are said to be *homeomorphic* or *topology equivalent*. If two spaces are homeomorphic then they are either both connected or both not connected. Any two closed intervals are homeomorphic so it makes sense to use the unit interval, $[0, 1]$, as the standard model. Similarly, any two half-open intervals are homeomorphic and any two open intervals are homeomorphic, but $[0, 1]$, $[0, 1)$, and $(0, 1)$ are pairwise non-homeomorphic. Indeed, suppose there is a homeomorphism $h : [0, 1] \to (0, 1)$. Removing the endpoint 0 from $[0, 1]$ leaves the closed interval connected, but removing $h(0)$ from $(0, 1)$ necessarily disconnects the open interval. Similarly the other two pairs are non-homeomorphic. The half-open interval can be continuously and bijectively mapped to the circle $\mathbb{S}^1 = \{x \in \mathbb{R}^2 \mid \|x\| = 1\}$. Indeed, $f : [0, 1) \to \mathbb{S}^1$ defined by $f(x) = (\sin 2\pi x, \cos 2\pi x)$ and illustrated in Figure I.7 is such a map. Removing the midpoint disconnects any one of the three intervals but

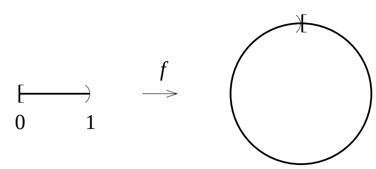

Figure I.7: A continuous bijection whose inverse is not continuous.

removing any one point from the circle keeps it connected. The circle is thus non-homeomorphic to the three intervals and represents a genuinely new type of curve.

To study curves extrinsically, we consider continuous maps of the four models. Primarily, we will use *paths*, $[0, 1] \to \mathbb{X}$, and *closed curves*, $\mathbb{S}^1 \to \mathbb{X}$. They are *simple* if the functions are injective.

Curves in the plane. Let $\gamma : \mathbb{S}^1 \to \mathbb{R}^2$ be a simple closed curve in the Euclidean plane. An interesting property is that it always decomposes the plane into two pieces, one on each side of the curve, as in Figure I.8 on the left.

JORDAN CURVE THEOREM. Let $\gamma : \mathbb{S}^1 \to \mathbb{R}^2$ be a simple closed curve. Then $\mathbb{R}^2 - \gamma(\mathbb{S}^1)$ has exactly two connected components, the bounded inside and the unbounded outside. The inside together with the image of the curve is a closed disk.

This result may seem obvious but all known proofs are cumbersome, and there are reasons to believe that there are no simple proofs. The fact that the inside together with the curve is homeomorphic to the closed disk, $\mathbb{B}^2 = \{x \in \mathbb{R}^2 \mid \|x\| \leq 1\}$, is known as the Schönflies Theorem. The Jordan Curve Theorem remains valid if we replace the plane by the sphere, $\mathbb{S}^2 = \{x \in \mathbb{R}^3 \mid \|x\| = 1\}$, but not if we replace it by the torus.

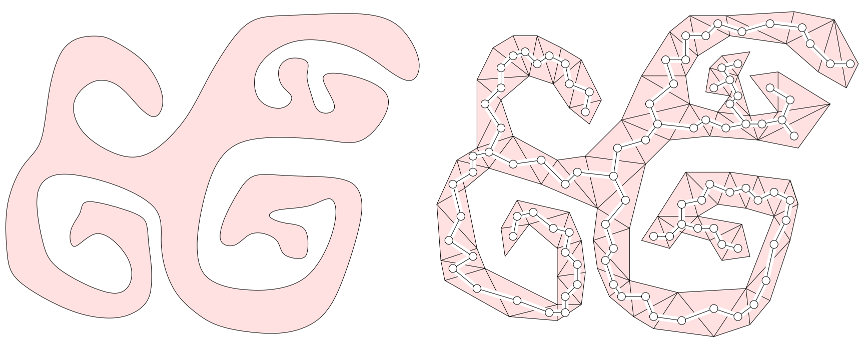

Figure I.8: Left: the shaded inside and the white outside of a closed curve in the plane. Right: a triangulation of a simple closed polygon approximating the closed curve and its dual tree.

If the image of $\gamma : \mathbb{S}^1 \to \mathbb{R}^2$ consists of finitely many line segments we often refer to it as a *closed polygon*. The computational problem of *triangulating* a simple closed polygon means we decompose the inside into triangles, as in Figure I.8 on the right. Usually, it is assumed that we do this without using any vertices other than the ones given by the polygon. It is not difficult to prove that such a triangulation always exists. The dual of such a triangulation is a tree, which implies that the number of diagonals is one less than the number of triangles. It also implies that every triangulation has an *ear*, a triangle bounded by one diagonal and two polygon edges. Letting $t(n)$ be the number of triangles used to triangulate a simple closed polygon with n vertices, we therefore have $t(n) = t(n-1) + 1 = n - 2$. It follows that the number of diagonals is $n - 3$, no matter how we triangulate the polygon.

Knots. When we embed a curve in three-dimensional space then we no longer decompose space in two pieces. A *knot* is an injective, continuous function $k : \mathbb{S}^1 \to \mathbb{R}^3$. Another knot ℓ is *equivalent* to k if it can be continuously deformed into k without crossing itself during this process. Equivalent knots are considered the same. The simplest knot is the *unknot*, also known as the

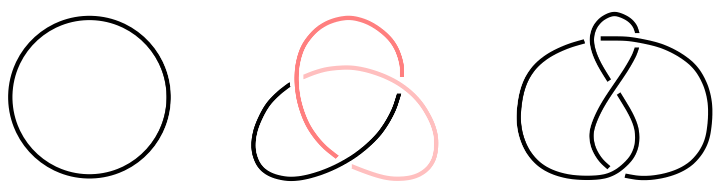

Figure I.9: From left to right: the unknot, the trefoil knot, and the figure-eight knot. The trefoil knot is tricolored.

trivial knot, which can be deformed to a geometric circle in $\mathbb{R}^3$. Two other rather easy knots are the *trefoil* and the *figure-eight knots*, both illustrated in Figure I.9. A subtlety in the definition of equivalent knots is that deformations in which knotted parts disappear in the limit are not allowed. It is therefore useful to think of knots as curves with small but positive thickness, similar to a physical string we may use to tie a knot.

Reidemeister moves. Let us follow a deformation of a knot by drawing its projections to a plane, keeping track of the under- and over-passes at crossings. We are primarily interested in generic projections defined by the absence of any violations to injectivity other than double-points where the curve crosses itself in the plane. In a generic deformation, we observe three types of non-generic projections that transition between generic projections, which are illustrated in Figure I.10. It is plausible and also true that any two generic projections of the same knot can be transformed into each other by Reidemeister moves.

It seems clear that the trefoil knot is not equivalent to the unknot, and there is indeed an elementary proof using Reidemeister moves. Call a piece of the knot from one under-pass to the next a *strand*. A *tricoloring* of a generic projection colors each strand with one of three colors such that

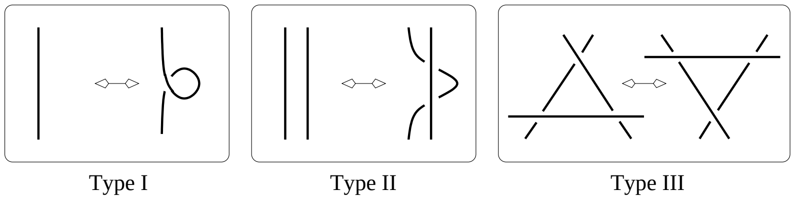

Figure I.10: The three types of Reidemeister moves.

(i) at each crossing either three colors or only one color come together;

(ii) at least two colors are used.

Figure I.9 shows that the standard projection of the trefoil knot is tricolorable. A useful property of Reidemeister moves is that they preserve tricolorability, that is, the projection before the move is tricolorable iff the projection after the move is tricolorable.

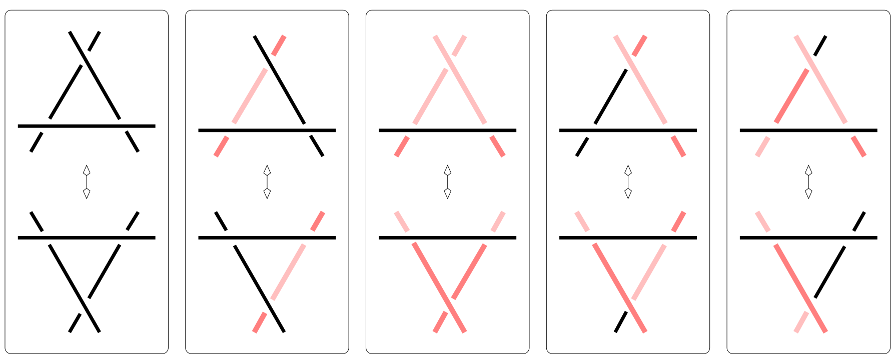

Figure I.11: The different cases in the proof that the Type III Reidemeister move preserves tricolorability.

Type I. Going from left to right in Figure I.10, we use the same one color, and going from right to left we observe that we have only one color coming together at the crossing.

Type II. From left to right we have two possibilities, either using only one color or going from two to three colors. The reverse direction is symmetric.

Type III. There are five cases to be checked, all shown in Figure I.11.

The trefoil knot is tricolorable and the unknot is not tricolorable. It follows that the two are not equivalent. It is not difficult to see that the figure-eight knot is not tricolorable. This implies that the trefoil knot and the figure-eight knot are different but the method is not powerful enough to distinguish the figure-eight from the unknot.

Links. A *link* is a collection of two or more disjoint knots. Equivalence between links is defined the same way as between knots, and Reidemeister moves again suffice to go from one generic projection to another. A disjoint plane *splits* a link if there are knots on both sides of the plane. A link is *splittable* if an equivalent link has a splitting plane. The *unlink* or *trivial link* of size two consists of two unknots that can be split, like the two circles in Figure I.12 on the left. The easiest non-splittable link consisting of two unknots is the *Hopf link*, which is shown in Figure I.12 in the middle. We can

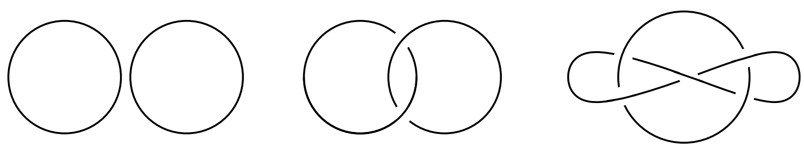

Figure I.12: From left to right: the unlink, the Hopf link, and the Whitehead link.

again use tricolorability to prove that the Hopf link is different from the unlink. Alternatively, we may count the crossings between the two knots, counting with a sign. Specifically, we orient each knot arbitrarily and we look at each crossing locally. If the under-pass goes from the left of the over-pass to its right then we count $+1$ and otherwise we count -1. The *linking number* is half the sum of the $+1$s and -1s over all crossings between the two knots. Changing the orientation of one knot but not the other has the effect of reversing the sign of the linking number. Clearly, Reidemeister moves do not affect the linking number. Since the linking number of the unlink is zero and that of the Hopf link is plus or minus one, we have another proof that the two links are different. An easy link that is not splittable but has vanishing linking number is the Whitehead link in Figure I.12. It consists of two unknots but cannot be tricolored, which again implies that it is not splittable.

Bibliographic notes. The Jordan Curve Theorem is well known also outside topology, in part because it seems so obvious but at the same time is difficult to prove. We refer to [3] for a deeper discussion of this result. Triangulations of simple closed polygons in the plane have been studied in computational geometry. Constructing such a triangulation in time proportional to the number of vertices seems rather difficult and the algorithm by Chazelle [2] that achieves this feat is not recommended for implementation. Knots and links have been studied for centuries and there are a number of good mathematical books on the subject, including the text by Adams [1].

[1] C. C. Adams. *The Knot Book.* W. H. Freeman, New York, 1994.

[2] B. Chazelle. Triangulating a simple polygon in linear time. *Discrete Comput. Geom.* **6** (1991), 485–524.

[3] C. T. C. Wall. *A Geometric Introduction to Topology.* Addison-Wesley, 1971.