II.3 Self-intersections

Since non-orientable compact 2-manifolds without boundary cannot be embedded in three-dimensional Euclidean space, all their models in that space occur with self-intersections. A practical motivation for looking at the phenomenon of self-intersections is to repair surface models of solid shapes.

Mapping into space. Let $\mathbb{M}$ be a 2-manifold and $f : \mathbb{M} \to \mathbb{R}^3$ a continuous mapping. For the time being assume f is smooth meaning derivatives of all orders exist. In the case at hand we have three real-valued functions in two variables. The matrix of partial derivatives is therefore

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial s_1} & \frac{\partial f_2}{\partial s_1} & \frac{\partial f_3}{\partial s_1} \\ \frac{\partial f_1}{\partial s_2} & \frac{\partial f_2}{\partial s_2} & \frac{\partial f_3}{\partial s_2} \end{bmatrix},$$

consisting of the three gradients. The rank of this matrix is at most two. The mapping f is an *immersion* of its derivative has full rank (rank 2) at every point of $\mathbb{M}$, and it is an *embedding* if f restricted to its image is a homeomorphism. For smooth mappings, there are three types of generic self-intersections, all illustrated in Figure II.9. The most interesting of the three is the branch

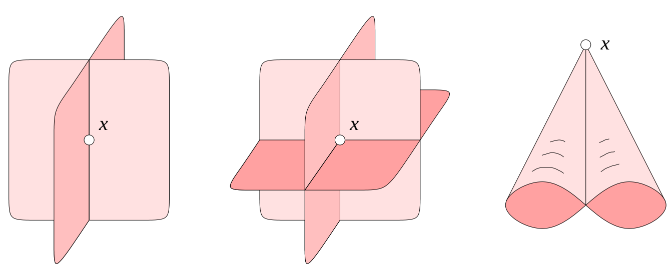

Figure II.9: From left to right: a double point, a triple point, a branch point.

point, which comes in several guises. We can construct it by cutting a disk from two sides toward the center, folding it, and re-glueing the sides as the self-intersection, as shown in Figure II.10.

Triangle meshes. Classifying the types of self-intersections is easier in the piecewise linear case in which $\mathbb{M}$ is given by a triangulation K. Since $\mathbb{M}$ is a 2-manifold, the triangles that contain a vertex form a disk, or perhaps half a

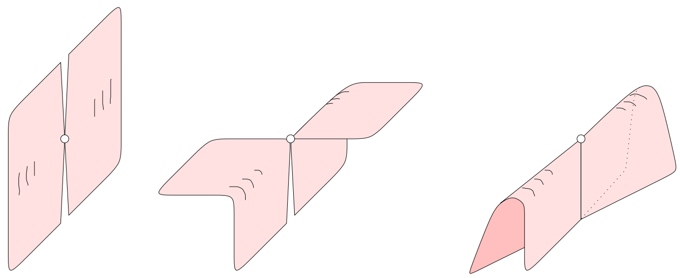

Figure II.10: Constructing the Whitney umbrella from a disk.

disk if $\mathbb{M}$ has boundary. It is not difficult to see that imposing this condition on the vertices suffices to guarantee that K triangulates a 2-manifold.

We put K into space by mapping each vertex to a point in $\mathbb{R}^3$. The edges and triangles are mapped to the convex hulls of the images of their vertices. This mapping is an embedding iff any two triangles are either disjoint or they share a vertex or they share an edge. Any other intersection is improper and referred to as a *crossing*. It is convenient to assume that the points are in general position, that is, no three are collinear and no four are coplanar. Under this assumption, there are only three types of crossing possible between two triangles, all shown in Figure II.11. Each crossing is a line segment common to two triangles. In

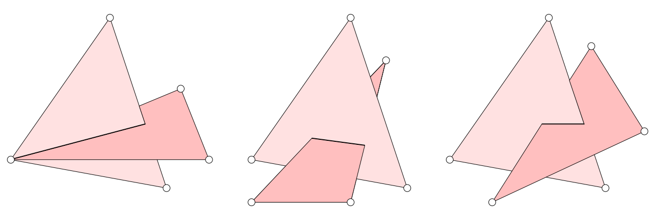

Figure II.11: The three ways two triangles in general position in $\mathbb{R}^3$ can cross each other.

the first case, one of the endpoints coincides with the image of a vertex. The other endpoint lies on a unique edge, there is a unique other triangle on the other side of that edge that continues the intersection. In the other two cases both endpoints lie on edges of the triangulation and the intersection has unique continuations in both directions.

Arcs and closed curves. Starting from a single crossing, we can trace the self-intersection in one or both directions, adding a line segment at a time. Since we have only finitely many triangles and thus finitely many line segments, each curve must either close up or end. In the first case we get a closed curve of almost all double points. Its preimage in K is either a pair of loops or a single loop that covers the closed curve twice. Such a double covering loop is necessarily orientation reversing hence $\mathbb{M}$ must have been non-orientable. To construct an example, sweep a line segment along a circle in $\mathbb{R}^3$. The line segment remains normal to the circle but its angle with the symmetry axis of the circle can change. If we take the angle from 0 to π during a full revolution then we get the Möbius strip. If we take it from 0 to $\frac{\pi}{2}$ we need a second full revolution before the surface is complete. We thus get a Möbius strip whose mapping to $\mathbb{R}^3$ crosses itself along the center circle, which is covered twice.

We conclude this section with two immersions of the Klein bottle in $\mathbb{R}^3$. In the first and perhaps most commonly known model, the neck of the bottle extends and bends backward, like a Flamingo, but then continues and passes through the surface. The closed intersection curve is the common image of two orientation preserving loops. To construct the second model, we sweep a pair of line segments along a circle in $\mathbb{R}^3$. The two line segments cross each other orthogonally at their respective midpoints and they are both orthogonal to the circle. During a full revolution we take the angle one line segment forms with the symmetry axis from 0 to π. Correspondingly, the angle formed by the other line segment goes from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$. The two line segments thus sweep out two Möbius strips crossing each other along their center circles. We can now complete the Klein bottle by connecting the two boundary curves by a circular arc, which we again sweep twice around the axis. In other words, we get the Klein bottle by sweeping a figure-8 curve along the circle, rotating it half-way so that after a full revolution the two lobes are exchanged. We now have an immersion in which intersection is a closed curve whose preimage consists of two orientation-reversing loops.

Bibliographic notes. The way surfaces mapped into $\mathbb{R}^3$ intersect is discussed in length and with many illustrations by Carter [2]. In the generic case such a mapping has only three types of singularities, double points, triple points, and branch points. Whitney proved that every d-manifold has an immersion in $\mathbb{R}^{2d-1}$ [4]. This implies that every 2-manifold can be immersed in $\mathbb{R}^3$, meaning there are smooth mappings without branch points. For the projective plane we must have a branch point or a triple point which implies that every immersion has a triple point [1]. Whitney also proved that every d-manifold can be embedded in $\mathbb{R}^{2d}$ [3], so every 2-manifold can be embedded in $\mathbb{R}^4$.

[1] T. F. BANCHOFF. Triple points and surgery of immersed surfaces. *Proc. Amer. Math. Soc.* **46** (1974), 403–413.

[2] J. S. CARTER. *How Surfaces Intersect in Space. An Introduction to Topology.* Second edition, World Scientific, Singapore, 1995.

[3] H. WHITNEY. The self-intersections of a smooth n-manifold in $2n$-space. *Annals of Math.* **45** (1944), 220–246.

[4] H. WHITNEY. The singularities of a smooth n-manifold in $(2n-1)$-space. *Annals of Math.* **45** (1944), 247–293.