IV.2 Homology

Similar to the fundamental group, the homology groups provide a language to talk about holes of topological spaces. The main differences are that homology groups are abelian, have fast algorithms, and easily extend to higher dimensions.

Cycles and holes. It has been said that in d-dimensional space the prisons are made of $(d-1)$-dimensional walls. This is because a wall of dimension $d-2$ or less cannot separate any portion of the space from the rest. A more formal expression of this observation is a statement that relates a subset of space with the complement using the language of homology.

ALEXANDER DUALITY THEOREM. For any non-empty, triangulable subset $\mathbb{X} \subseteq \mathbb{S}^d$ we have $\tilde{\mathsf{H}}_p(\mathbb{X}) \simeq \tilde{\mathsf{H}}^{d-p-1}(\mathbb{S}^d - \mathbb{X})$.

In words, the p-th reduced homology group of $\mathbb{X}$ is isomorphic to the $(d-p-1)$-st reduced cohomology group of its complement in $\mathbb{S}^d$. More intuitively, the p-dimensional cycles in $\mathbb{X}$ surround the $(d-p-1)$-dimensional holes. It will take a while before all terms in the theorem will be defined. There are, however, valuable messages we can take away without penetrating the full depths of the theorem. The first is the existence of a direct relationship between the cycles and the holes. The second is that this relationship requires a fixed ambient space, in this case the d-dimensional sphere which is a good model of d-dimensional Euclidean space. In homology theory, we gain generality by almost exclusively talking about cycles and thus not committing to any ambient space.

Chain complexes. Let K be a simplicial complex. A *p-chain* is a formal sum of p-simplices in K. The standard notation for this formal sum is $c = \sum a_i \sigma_i$, where σ_i is a p-simplices in K and a_i is either 0 or 1. Alternatively, we can think of c as the set of p-simplices σ_i with $a_i = 1$, but this would make generalizations to coefficient groups other than addition modulo 2 more cumbersome. Two p-chains are added componentwise, like polynomials. Specifically, if $c_0 = \sum b_i \sigma_i$ then $c + c_0 = \sum (a_i + b_i)\sigma_i$, where the coefficients are integers modulo 2, so $1+1 = 0$. In set notation, the sum of two p-chains is their symmetric difference. The p-chains together with the addition operation form the *group of p-chains* denoted as $(\mathsf{C}_p, +)$ or simply C_p if the operation is understood. Associativity follows from associativity of addition modulo 2. The neutral element is $0 =$

$\sum 0\sigma_i$. The inverse of c is $-c = c$ since $c+c = 0$. Finally, C_p is abelian because addition modulo 2 is commutative.

We have a group of p-chains for each integer p. For $p < 0$ and $p > \dim K$ this group is trivial, consisting only of the neutral element. To relate these groups, we define the *boundary* of a p-simplex as the sum of its $(p-1)$-dimensional faces. Writing $\sigma = [u_0, u_i, \ldots, u_p]$ for the simplex spanned by the vertices u_0 to u_p, its boundary is

$$\partial_p \sigma \quad = \quad \sum_{i=0}^{p} [u_0, \ldots, \hat{u}_i, \ldots, u_p],$$

where the hat indicates that u_i be dropped. For a p-chain $c = \sum a_i \sigma_i$ the boundary is the sum of boundaries of its simplices, $\partial_p c = \sum a_i \partial_p \sigma_i$. Taking the boundary commutes with addition, $\partial_p(c + c_0) = \partial_p c + \partial_p c_0$. Hence, $\partial_p : \mathsf{C}_p \to \mathsf{C}_{p-1}$ is a homomorphism. The *chain complex* is the sequence of chain groups connected by boundary homomorphisms,

$$\ldots \stackrel{\partial_{p+2}}{\to} \mathsf{C}_{p+1} \stackrel{\partial_{p+1}}{\to} \mathsf{C}_p \stackrel{\partial_p}{\to} \mathsf{C}_{p-1} \stackrel{\partial_{p-1}}{\to} \ldots$$

It will be convenient to drop the index from the boundary homomorphism since it is implied by the dimension of the chain.

Cycles and boundaries. We distinguish two particular types of chains and use them to define homology groups. A *p-cycle* is a p-chain with empty boundary, $\partial c = 0$. Since ∂ commutes with addition, we have a *group of p-cycles*, denoted as $\mathsf{Z}_p \leq \mathsf{C}_p$, which is a subgroup of the group of p-chains. In other words, the group of p-cycles is the kernel of the p-th boundary homomorphism, $\mathsf{Z}_p = \ker \partial_p$. Since the chain groups are abelian so are their cycle subgroups. Consider $p = 0$ as an example. The boundary of every vertex is zero, and 0 is indeed the only element in C_{-1}. Hence, $\mathsf{Z}_0 = \ker \partial_0 = \mathsf{C}_0$.

A *p-boundary* is a p-chain that is the boundary of a $(p+1)$-chain, $c = \partial d$ with $d \in \mathsf{C}_{p+1}$. Since ∂ commutes with addition, we have a *group of p-boundaries*, denotes as $\mathsf{B}_p \leq \mathsf{C}_p$. In other words, the group of p-boundaries is the image of the $(p+1)$-st boundary homomorphism, $\mathsf{B}_p = \operatorname{im} \partial_{p+1}$. Since the chain groups are abelian so are their boundary subgroups. Consider $p = 0$ as an example. Every 1-chain consists of some number of edges with twice as many endpoints. Taking the boundary cancels duplicate endpoints in pairs leaving and even number. Hence, every 0-chain with an even number of vertices in each component is a 0-boundary. If K is connected this implies that half the 0-cycles are 0-boundaries. The fundamental property that makes homology work is that the boundary of a boundary is necessarily zero.

FUNDAMENTAL LEMMA OF HOMOLOGY. $\partial_p \partial_{p+1} d = 0$ for every integer p and every $(p+1)$-chain d.

PROOF. We just need to show that $\partial_p \partial_{p+1} \tau = 0$ for a $(p+1)$-simplex τ. The boundary, $\partial_{p+1}\tau$, consists of all p-faces of τ. Every $(p-1)$-face of τ belongs to exactly two p-faces, so $\partial_p(\partial_{p+1}\tau) = 0$. ◻

It follows that every p-boundary is also a p-cycle or, equivalently, that B_p is a subgroup of Z_p. Figure IV.4 illustrates the subgroup relations among the three types of groups and their connection across dimensions through boundary homomorphisms.

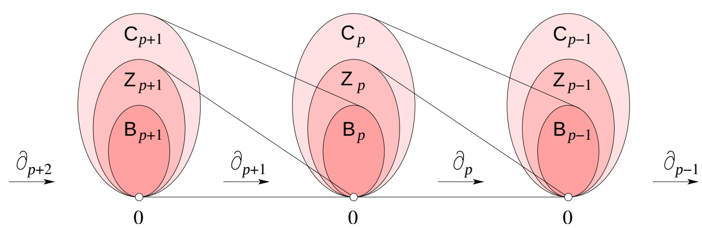

Figure IV.4: The chain complex consisting of a linear sequence of chain, cycle, and boundary groups connected by boundary homomorphisms.

Homology groups. Since the boundaries form subgroups of the cycle groups, we can take quotients, which are the *homology groups*, $\mathsf{H}_p = \mathsf{Z}_p/\mathsf{B}_p$. Each element is a collection obtained by adding each p-boundary to a given p-cycle, $c + \mathsf{B}_p$ with $c \in \mathsf{Z}_p$. More formally, this collection is called a *coset*. Any two cycles in the same coset are said to be *homologous*, which is denoted as $c_1 \sim c_2$; see Figure IV.5. We may take c as the representative of this coset

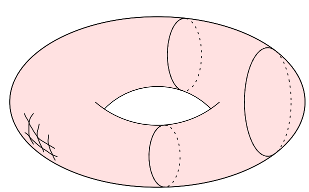

Figure IV.5: A torus with three closed curves, each a 1-cycle. Only one a 1-boundary and it is homologous to the sum of the other two. The sum of the three curves is therefore a 1-boundary, namely of the pair of pants between them.

but any other cycle of the form $c + \partial d$ does as well. Similarly, addition of two cosets, $(c + \mathsf{B}_p) + (c_0 + \mathsf{B}_p) = (c + c_0) + \mathsf{B}_p$, is independent of the representatives and therefore well defined. We thus see that H_p is indeed a group, and because Z_p is abelian so is H_p.

For groups the cardinality is called the *order*. Since we use addition modulo 2, $\operatorname{ord} \mathsf{C}_p = 2^{n_p}$ if n_p is the number of p-simplices in K, and C_p is isomorphic to $\mathbb{Z}_2^{n_p}$, the group of bit-vectors of length n_p together with the exclusive-or operation. This is an n_p-dimensional vector space which is therefore generated by n_p bit-vectors, for example the vectors that have a single 1 each corresponding to individual p-simplices in K. The dimension is referred to as the *rank* of the vector space, $n_p = \operatorname{rank} \mathbb{Z}_2^{n_p} = \operatorname{rank} \mathsf{C}_p$. The cycles, boundaries, and cochains exhibit the same vector space structure, except that their dimension is usually less than that of the chains. The number of cycles in a coset is the order of B_p, hence the number of cosets in the homology group is $\operatorname{ord} \mathsf{H}_p = \operatorname{ord} \mathsf{Z}_p / \operatorname{ord} \mathsf{B}_p$. Equivalently, the rank is the difference,

$$\beta_p \quad = \quad \operatorname{rank} \mathsf{H}_p \quad = \quad \operatorname{rank} \mathsf{Z}_p - \operatorname{rank} \mathsf{B}_p.$$

This is the *p-th Betti number* of K. This discussion suggests two ways to illustrate a homology group, as a partition of the set of cycles into cosets and the hypercube of dimension β_p; see Figure IV.6. As an example consider the triangulation of a torus. There are only four cosets in H_1, namely B_1, $a + B_1$, $b + b_1$, and $(a + b) + B_1$, where a and b are the non-bounding 1-cycles that go once around the hole and the arm of the torus. The two corresponding cosets, $a + B_1$ and $b + B_1$, generate the first homology group.

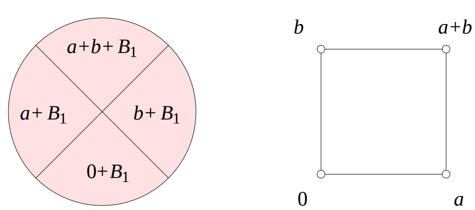

Figure IV.6: The first homology group of the torus has rank 2 and order 4. On the left, the four elements are cosets in the group of 1-cycles. On the right, the four elements are the vertices of a square.

Euler-Poincaré formula. Recall that the Euler characteristic of a simplicial complex is the alternating count of simplices. Writing $n_p = \operatorname{rank} \mathsf{C}_p$ for the number of p-simplices in K, as before, and $z_p = \operatorname{rank} \mathsf{Z}_p$ and $b_p = \operatorname{rank} \mathsf{B}_p$, we have $n_p = z_p + b_{p-1}$. The Euler characteristic is the alternating sum of the n_p, which is therefore

$$\begin{aligned}\chi &= \sum_{p\geq 0}(-1)^p(z_p + b_{p-1}) \\ &= \sum_{p\geq 0}(-1)^p(z_p - b_p) \\ &= \sum_{p\geq 0}(-1)^p\beta_p.\end{aligned}$$

To appreciate the beauty of this result we need to know that homology groups do not depend on the triangulation chosen for a topological space. The technical proof of this claim is a bit cumbersome and omitted but even the more general result that homotopy equivalent spaces have isomorphic homology groups is plausible. For example, we can free ourselves from the triangulation entirely and define chains in terms of continuous maps from the standard simplex into the space $\mathbb{X}$. This gives rise to so-called singular homology, which has been shown to give groups isomorphic to the ones we get by simplicial homology, which is the theory we describe in this section. If we now have a continuous map $f : \mathbb{X} \to \mathbb{Y}$ we can literally map the cycles from $\mathbb{X}$ to $\mathbb{Y}$. If f is a homotopy equivalence then we can map both ways and thus guarantee isomorphic homology groups. This also implies that the Euler characteristic is an invariant of the space, that is, it does not depend on the simplicial complex we use to triangulate it.

Euler-Poincaré Theorem. The Euler characteristic of a topological space is the alternating sum of its Betti numbers, $\chi = \sum_{p\geq 0}\beta_p$.

As an example consider the torus. It is connected so half of its 0-cycles are 0-boundaries implying $\text{ord}\,\mathsf{Z}_0/\text{ord}\,\mathsf{B}_0 = 2$ and therefore $\beta_0 = 1$. We have seen that $\beta_1 = 2$. Finally, $\beta_2 = 1$ because there is only one 2-cycle, namely the sum of all triangles, and no 2-boundary. The Euler characteristic of the torus is indeed $\chi = 1 - 2 + 1 = 0$.

Reduced homology. We obtain a small but often useful modification of homology by adding the *augmentation map* $\epsilon : \mathsf{C}_0 \to \mathbb{Z}_2$ defined by $\epsilon(u) = 1$ for each vertex u. We thus get

$$\ldots \xrightarrow{\partial_2} \mathsf{C}_1 \xrightarrow{\partial_1} \mathsf{C}_0 \xrightarrow{\epsilon} \mathbb{Z}_2 \xrightarrow{0} 0 \to \ldots$$

Cycles and boundaries are defined as before and the only difference we notice is for Z_0 which now requires that each 0-cycle has an even number of vertices.

This results in the *reduced homology groups*, $\tilde{\mathsf{H}}_p$, and the *reduced Betti numbers*, $\tilde{\beta}_p = \operatorname{rank} \tilde{\mathsf{H}}_p$. Assuming K is non-empty, we have $\tilde{\beta}_p = \beta_p$ for all $p \geq 1$ and $\tilde{\beta}_0 = \beta_0 - 1$. For $K = \emptyset$ we have $\tilde{\beta}_{-1} = 1$ since both elements of $\mathbb{Z}_2$ are (-1)-cycles (they belong to the kernel) but only one is a (-1)-boundary (it belongs to the image of the augmentation map).

Degree of a map. We can use homology to prove Brouwer's Fixed Point Theorem, now in the general, d-dimensional setting. To this end let $\varphi : \mathbb{S}^d \to \mathbb{S}^d$ be a continuous map. Let c be the unique generator of the d-th homology group. Then $\varphi(c)$ is either homologous to c or to 0. In other words, $\varphi(c) \sim \delta c$ and $\delta \in \{0, 1\}$ is called the *degree* of φ. If φ is the identity then $\delta = 1$. However, if φ extends a continuous map $\psi : \mathbb{B}^{d+1} \to \mathbb{S}^d$ then the induced map on homology groups $\varphi_* : \mathsf{H}_d(\mathbb{S}^d) \to \mathsf{H}_d(\mathbb{S}^d)$ is the composite of two induced maps,

$$\mathsf{H}_d(\mathbb{S}^d) \to \mathsf{H}_d(\mathbb{B}^{d+1}) \to \mathsf{H}_d(\mathbb{S}^d),$$

where the first is induced by inclusion. The middle group is trivial, hence $\delta = 0$. We are now ready to prove the theorem.

Brouwer's Fixed Point Theorem. A continuous map $f : \mathbb{B}^{d+1} \to \mathbb{B}^{d+1}$ has at least one fixed point $x = f(x)$.

Proof. Let $A, B : \mathbb{S}^d \to \mathbb{S}^d$ be maps defined by $A(x) = (x - f(x))/\|x - f(x)\|$ and $B(x) = x$. B is the identity and therefore has degree 1. If f has no fixed point then A is well defined and has degree 0 because it extends a map from the ball to the sphere. We now construct $H : \mathbb{S}^d \times [0, 1] \to \mathbb{S}^d$ defined by $H(x, t) = (x - tf(x))/\|x - tf(x)\|$. For $t = 1$ we have $x \neq f(x)$ because there is no fixed point and for $t < 1$ we have $x \neq tf(x)$ because $\|x\| = 1 > \|tf(x)\|$. We conclude that H is a homotopy between A and B which implies that the degrees of the two are the same, a contradiction. ◻

Bibliographic notes. Like many other concepts in topology, homology groups have been introduced by Henri Poincaré in one of a series of papers on "analysis situ" [4]. He named the ranks of the homology groups after another mathematician, Betti, who introduced a slightly different version years earlier. The field experienced a rapid development during the twentieth century. There were many competing theories, simplicial and singular homology just being two examples, which have been consolidated by axiomizing the assumptions under which homology groups exist [1]. Today we have a number of well established textbooks in the field. We refer to Giblin [2] for an intuitive introduction and to Munkres [3] for a more comprehensive source..

[1] S. Eilenberg and N. Steenrod. *Foundations of Algebraic Topology.* Princeton Univ. Press, New Jersey, 1952.

[2] P. J. Giblin. *Graphs, Surfaces and Homology.* Chapman and Hall, London, 1981.

[3] J. R. Munkres. *Elements of Algebraic Topology.* Addison-Wesley, Redwood City, California, 1984.

[4] H. Poincaré. Complément à l'analysis situs. *Rendiconti del Circolo Matematico di Palermo* **13** (1899), 285–343. .