IV.4 Cohomology

In this section, we introduce cohomology groups. They are similar to homology groups but less geometric and motivated primarily by algebraic considerations.

Groups of maps. Let $G = \mathbb{Z}_2$, the group of two elements, 0 and 1, together with addition modulo 2. All abelian groups we have encountered so far are isomorphic to G^n for some finite integer n. Let A be such a group and $\varphi : A \to G$ a homomorphism. It suffices to specify φ for the generators of A. Letting $\varphi_0 : A \to G$ be another homomorphism, the sum of the two is defined by $(\varphi + \varphi_0)(a) = \varphi(a) + \varphi_0(a)$. This is again a homomorphism because

$$\begin{aligned}(\varphi + \varphi_0)(a+b) &= \varphi(a+b) + \varphi_0(a+b) \\ &= \varphi(a) + \varphi(b) + \varphi_0(a) + \varphi_0(b) \\ &= (\varphi + \varphi_0)(a) + (\varphi + \varphi_0)(b).\end{aligned}$$

We therefore have a *group of homomorphisms* from A to G, denoted as $\mathrm{Hom}(A, G)$. If A is isomorphic to G^n then so is $\mathrm{Hom}(A, G)$. Given another group B and a homomorphism $f : A \to B$, there is a *dual homomorphism*, $\tilde{f} : \mathrm{Hom}(B, G) \to \mathrm{Hom}(A, G)$ that maps $\psi : B \to G$ to the composite $\tilde{f}(\psi) = \psi \circ f : A \to B \to G$. The map $\tilde{f}$ is indeed a homomorphism since

$$\begin{aligned}\tilde{f}(\psi + \psi_0)(b) &= (\psi + \psi_0) \circ f(b)) \\ &= \psi(f(b)) + \psi_0(f(b)) \\ &= \tilde{f}(\psi)(b) + \tilde{f}(\psi_0)(b)\end{aligned}$$

for every $b \in B$. The group of homomorphisms and the dual homomorphism can be defined for more general abelian groups A, B, and G but this will not be necessary for our purposes.

Simplicial cohomology. Let K be a simplicial complex. We construct cohomology groups by turning chain groups into groups of homomorphisms and boundary operators into their dual homomorphisms. To begin, we define a *p-dimensional cochain* as a homomorphism $\varphi : \mathsf{C}_p \to G$, where $G = \mathbb{Z}_2$ as before. Given a p-chain $c \in \mathsf{C}_p$, the cochain evaluates c by mapping it to 0 or 1. It is common to write this evaluation like a scalar product, $\varphi(c) = \langle \varphi, c \rangle$. Letting ℓ be the number of p-simplices $\sigma \in c$ with $\varphi(\sigma) = 1$, we have $\langle \varphi, c \rangle = 1$ iff ℓ is odd. Considering chains and cochains as sets, the evaluation thus distinguishes odd from even intersections.

The p-dimensional cochains form the *group of p-cochains*, $\mathsf{C}^p = \mathrm{Hom}(\mathsf{C}_p, G)$. Recall that the boundary operator is a homomorphism $\partial_p : \mathsf{C}_p \to \mathsf{C}_{p-1}$. It thus defines a dual homomorphism, the *coboundary operator*

$$\delta^{p-1} : \mathrm{Hom}(\mathsf{C}_{p-1}, G) \to \mathrm{Hom}(\mathsf{C}_p, G),$$

or simply $\delta : \mathsf{C}^{p-1} \to \mathsf{C}^p$. Let φ be a $(p-1)$-cochain and ∂c a $(p-1)$-chain. By definition of dual homomorphism, φ applied to ∂c is the same as $\delta\varphi$ applied to c, $\langle \varphi, \partial c\rangle = \langle \delta\varphi, c\rangle$. Suppose for example that φ evaluates a single $(p-1)$-simplex to 1 and all others to 0. Then $\delta\varphi$ evaluates all p-dimensional cofaces to 1 and all others to 0. Since the coboundary operator runs in a direction opposite to the boundary operator it raises the dimension. Its kernel is the *group of cocycles* and its image is the *group of coboundaries*,

$$\begin{aligned} \mathsf{Z}^p &= \ker \delta^p : \mathsf{C}^p \to \mathsf{C}^{p+1}, \\ \mathsf{B}^p &= \mathrm{im}\, \delta^{p+1} : \mathsf{C}^{p+1} \to \mathsf{C}^p. \end{aligned}$$

Recall the Fundamental Lemma of Homology which says that $\partial \circ \partial : \mathsf{C}_{p+1} \to \mathsf{C}_{p-1}$ is the zero homomorphism. We therefore have $\langle \delta \circ \delta(\varphi), c\rangle = \langle \delta(\varphi), \partial(c)\rangle = \langle \varphi, \partial \circ \partial(c)\rangle = 0$. In other words, $\delta \circ \delta : \mathsf{C}^{p-1} \to \mathsf{C}^{p+1}$ is also the zero homomorphism. Hence the coboundary groups are subgroups of the cocycle groups and we have the familiar picture, except that the maps now go from right to left, as in Figure IV.13.

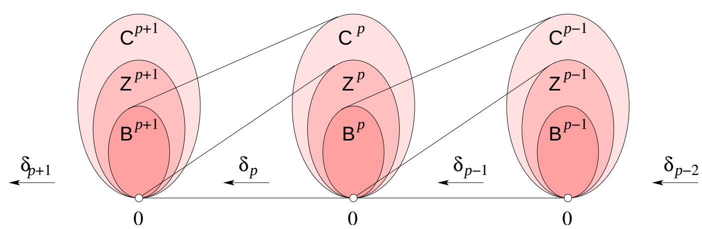

Figure IV.13: The cochain complex consisting of a linear sequence of cochain, cocycle, and coboundary groups connected by coboundary homomorphisms.

Definition. The *p-th cohomology group* is the quotient of p-cocycle modulo p-coboundary groups, $\mathsf{H}^p = \mathsf{Z}^p/\mathsf{B}^p$, for all p.

An example. To get a better feeling for cohomology let us consider the triangulation of the annulus in Figure IV.14. The 0-cochain that evaluates every single vertex to 1 is a 0-cocycle because every edge has exactly two vertices, which implies that the coboundary of the particular 0-cochain is the zero homomorphism. This is the only non-trivial 0-cocycle, and since for dimensional reasons there are no non-trivial 0-coboundaries, this implies that the 0-cohomomology group, H^0, has rank 1. One dimension up we consider a 1-

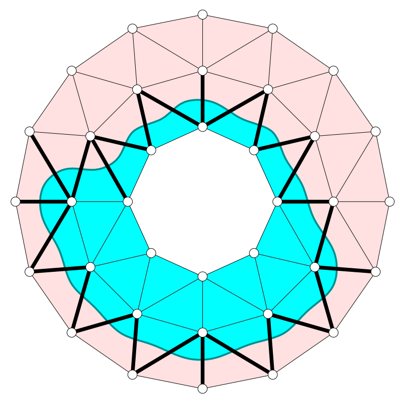

Figure IV.14: The 1-cocycle is drawn by highlighting the edges it evaluates to 1. They all cross the "dual" closed curve. The 1-cocycle is a 1-coboundary because it is the coboundary of the 0-cochain that evaluates a vertex to 1 iff it lies inside the closed curve.

cochain $\varphi : \mathsf{C}_1 \to G$. Its coboundary is the 2-chain $\delta\varphi : \mathsf{C}_2 \to G$ that evaluates a triangle to 1 iff it is the coface of an odd number of edges evaluated to 1 by φ. Hence φ is a 1-cocycle iff every triangle is incident to an even number of edges evaluating to 1. A 1-cocycle thus looks like a picket fence; see Figure IV.14. We can draw a closed curve such that an edge evaluates to 1 iff it crosses the curve. A 1-chain is therefore evaluated to the parity of the number of times it crosses that curve. If the 1-chain is a 1-cycle then this number is necessarily even. The 1-cocycle in Figure IV.14 is also a 1-coboundary. A 1-cocycle that is not the image of a 0-cochain is a picket fence that starts with an outer boundary edge and ends with an inner boundary edge. This is the only kind, hence the 1-cohomology group, H^1, has rank 1. For dimensional reasons every 2-cochain of the annulus is also a 2-cocycle. On the other hand, every collection of triangles is bounded by a collection of closed curves, so we can construct a dual picket fence, a 1-cochain that is also a 1-coboundary. It follows that the 2-cohomology group, H^2, has rank 0. Observe the these numbers are the same as the ranks of the corresponding homology groups. This is not a coincidence.

Coboundary matrix. Recall that we can get the rank of the p-th homology group from two boundary matrices transformed into normal form by row and column operations. As illustrated in Figure IV.15, the rank of H_p is the number of zero columns in the p-th matrix minus the number of non-zero rows in the $(p+1)$-st matrix. Recall that a cochain evaluates a p-simplex to 1 iff its

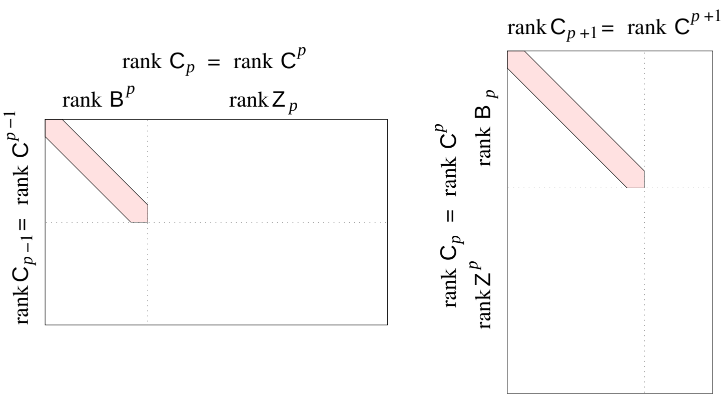

Figure IV.15: The p-th and $(p+1)$-st boundary matrices in normal form. They are also the corresponding coboundary matrices in normal form transposed.

coboundary evaluates each $(p+1)$-coface of this p-simplex to 1. It follows that the coboundary matrices are the boundary matrices transposed. The normal form of the boundary matrices thus already contains the information we need to get at the ranks of the cohomology groups. Specifically, $\operatorname{rank}\mathsf{H}^p = \operatorname{rank}\mathsf{Z}^p - \operatorname{rank}\mathsf{B}^p$. The rank of the cocycle group is the number of zero rows in the $(p+1)$-st boundary matrix and the rank of the coboundary group is the number of non-zero columns in the p-th boundary matrix, both in normal form. The number of columns of the p-th matrix is the number of rows of the $(p+1)$-st matrix, hence $\operatorname{rank}\mathsf{B}^p + \operatorname{rank}\mathsf{Z}_p = \operatorname{rank}\mathsf{Z}^p + \operatorname{rank}\mathsf{B}_p$. This implies

$$\begin{aligned}\operatorname{rank}\mathsf{H}^p &= \operatorname{rank}\mathsf{Z}^p - \operatorname{rank}\mathsf{B}^p \\ &= \operatorname{rank}\mathsf{Z}_p - \operatorname{rank}\mathsf{B}_p \;=\; \operatorname{rank}\mathsf{H}_p.\end{aligned}$$

For modulo 2 arithmetic the rank determines the group, hence homology and cohomology groups are isomorphic, $\mathsf{H}_p \simeq \mathsf{H}^p$ for all p.

Block decomposition. For manifolds there are additional and more interesting relationships between homology and cohomology. Let K be a triangulation of a d-manifold. Recall that the barycentric subdivision, $\mathrm{Sd}\,K$, is obtained by connecting the barycenters of the simplices in K, as illustrated in Figure IV.16. Label each vertex in $\mathrm{Sd}K$ by the dimension of the corresponding simplex in K and note that the vertices have distinct labels if they belong to the same simplex in the subdivision. The vertex with smallest label is unique. Letting u be the barycenter of $\sigma \in K$, the *block dual* to σ is the union of interiors of simplices in $\mathrm{Sd}K$ for which u is the vertex with minimum label; see Figure IV.16. In the case of a 3-manifold, the block dual to a vertex, edge, triangle,

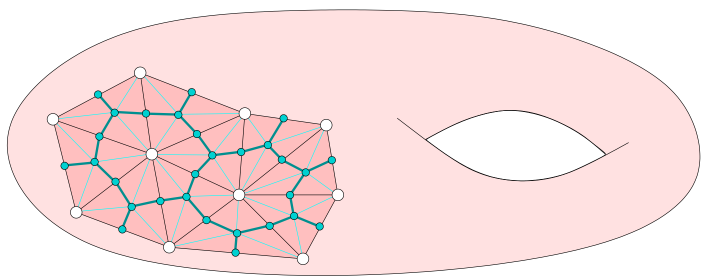

Figure IV.16: A small piece of a triangulation of the torus, the barycentric subdivision, and the dual block decomposition.

and tetrahedron is an open ball, an open disk, and open interval, and a point. The relationship between K and its dual block decomposition is much like that between the Delaunay triangulation and its dual Voronoi diagram.

Let $\sigma \in K$ with $\dim \sigma = j$. The dimension of the dual block is $d - j$. The alternating sum of simplices in $\mathrm{Sd}K$ that constitute the dual block is one minus the Euler characteristic of the $(d-j-1)$-dimensional sphere,

$$\begin{aligned} 1 - \chi(\mathbb{S}^{d-j-1}) &= 1 - (1 + (-1)^{d-j-1}) \\ &= (-1)^{d-j}. \end{aligned}$$

It follows that the Euler characteristic of K is the alternating sum of blocks. The number of $(d-j)$-dimensional blocks is n_j, so we get

$$\chi(K) = \sum_{j=0}^{d} (-1)^j n_j = \sum_{j=0}^{d} (-1)^j n_{d-j}.$$

For odd d this implies $\chi(K) = -\chi(K)$ which can only be true if the Euler characteristic vanishes. This is rather disappointing. After playing such an important role in the classification of 2-manifolds we find that the Euler characteristic says nothing about the type of a 3-manifold.

Duality theorems. It is tempting to think that the block dual to a j-simplex in the triangulation of a d-manifold is necessarily an open, $(d-j)$-dimensional ball. Equivalently, the link of the j-simplex is a $(d-j-1)$-sphere. This is not true but counterexamples are difficult to construct. However, there is a weaker property that is true and suffices for the correctness of the above argument on Euler characteristics. Letting D be a $(d-j)$-dimensional block, we write $\bar{D}$ for its closure and $\dot{D} = \bar{D} - D$ for its boundary. Then the relative homology of the pair $(\bar{D}, \dot{D})$ is that of the $(d-j)$-dimensional ball relative its boundary, namely $\mathsf{H}_p(\bar{D}, \dot{D}) \simeq \mathbb{Z}_2$ if $p = d-j$ and it vanishes if $p \neq d-j$. This property of blocks can be used to prove the following striking symmetry of manifolds.

POINCARÉ DUALITY THEOREM. Let $\mathbb{M}$ be a compact, triangulated d-manifold. Then $\mathsf{H}_p(\mathbb{M}) \simeq \mathsf{H}^{d-p}(\mathbb{M})$ for all p.

Together with $\mathsf{H}_p(\mathbb{M}) \simeq \mathsf{H}^p(\mathbb{M})$ this implies $\beta_j = \beta_{d-j}$ for all j. By the Euler-Poincaré Theorem, the Euler characteristic is the alternating sum of Betti numbers. This implies again that the Euler characteristic vanishes if the dimension of the manifold is odd. A generalization to manifolds with boundary explains the similarity between the non-cobounding cocycles and the non-bounding relative cycles in Figure IV.14.

LEFSCHETZ DUALITY THEOREM. Let $\mathbb{M}$ be a compact, triangulated d-manifold with boundary. Then $\mathsf{H}_p(\mathbb{M}, \operatorname{bd}\mathbb{M}) \simeq \mathsf{H}^{d-p}(\mathbb{M})$ for all p.

Bibliographic notes. Similar to homology, cohomology is an established topic within algebraic topology today. The question whether the link of a j-simplex in a triangulation of a d-manifold is necessarily a $(d-j-1)$-sphere had been open for a while until Robert Edwards established that this is not the case [1]. The duality theorems stated in this section have originally been proven in more elementary form before being reformulated in terms of homology and cohomology [2].

[1] R. D. EDWARDS. Approximating certain cell-like maps by homeomorphisms. *Notices Amer. Math. Soc.* **24** (1977), A647.

[2] J. R. MUNKRES. *Elements of Algebraic Topology.* Addison-Wesley, Redwood City, California, 1984.