V.2 Transversality Condition

Given a Morse function, we can follow the gradient flow and decompose the manifold depending on where the flow originates and where it ends. To recover the homology of the manifold from this decomposition, we require that the function satisfies an additional genericity assumption.

Integral lines. Recall the 1-parameter group of diffeomorphisms $\varphi : \mathbb{R} \times \mathbb{M} \to \mathbb{M}$ defined by a Morse function f on a manifold $\mathbb{M}$ with a Riemannian metric. The *integral line* that passes through a regular point $x \in \mathbb{M}$ is $\gamma = \gamma_x : \mathbb{R} \to \mathbb{M}$ defined by $\gamma(t) = \varphi(t, x)$; see Figure V.5. It is the solution to

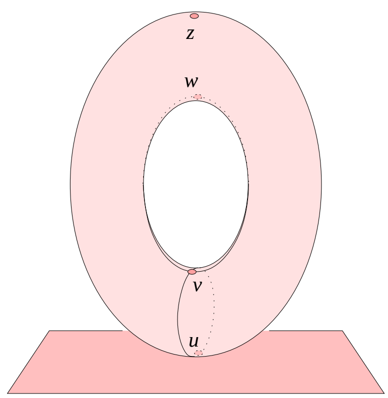

Figure V.5: The upright torus with the four integral lines that end at the two saddles.

the ordinary differential equation defined by $\dot{\gamma}(t) = \nabla f(\gamma(t))$ and the initial condition $\gamma(0) = x$. Because φ and therefore γ are defined for all $t \in \mathbb{R}$, the integral line necessarily approaches a critical point, both for t going to plus and to minus infinity. We call these critical points the *origin* and the *destination* of the integral line,

$$\begin{aligned} \text{org}(\gamma) &= \lim_{t \to -\infty} \gamma(t); \\ \text{dest}(\gamma) &= \lim_{t \to \infty} \gamma(t). \end{aligned}$$

The function increases along the integral line which implies that $\text{org}(\gamma) \neq \text{dest}(\gamma)$. The Existence and Uniqueness Theorems of ordinary differential equations imply that the integral line that passes through another regular point y is either disjoint from or the same as the one passing through x, $\text{im}\, \gamma_x = \text{im}\, \gamma_y$

or $\text{im}\,\gamma_x \cap \text{im}\,\gamma_y = \emptyset$. This property suggests we decompose the manifold into integral lines or unions of integral lines with shared characteristics.

Stable and unstable manifolds. The *stable manifold* of a critical point u of f is the point itself together with all regular points whose integral lines end at u. Symmetrically, the *unstable manifold* of u is the point itself together with all regular points whose integral lines originate at u. More formally,

$$\begin{aligned} S(u) &= \{u\} \cup \{x \in \mathbb{M} \mid \text{dest}(\gamma_x) = u\}; \\ U(u) &= \{u\} \cup \{y \in \mathbb{M} \mid \text{org}(\gamma_y) = u\}. \end{aligned}$$

The function increases along integral lines. It follows that $f(u) \geq f(x)$ for all points x in the stable manifold of u. This is the reason why $S(u)$ is sometimes referred to as the *descending manifold* of u. Symmetrically, $f(u) \leq f(y)$ for all points y in the unstable manifold of u and $U(u)$ is sometimes referred to as the *ascending manifold* of u.

Suppose the dimension of $\mathbb{M}$ is d and the index of the critical point u is p. Then there is a $(p-1)$-sphere of directions along which integral lines approach u. It can be proved that together with u these integral lines form an open ball of dimension p and that $S(u)$ is a submanifold homeomorphic to $\mathbb{R}^p$ that is immersed in $\mathbb{M}$. It is not embedded because distant points in $\mathbb{R}^p$ may map to arbitrarily close points in $\mathbb{M}$, as we can see in Figure V.5. For example, the saddle v has a stable 1-manifold consisting of two integral lines that merge at v to form one open, connected interval. The two ends of the interval approach the minimum, u, which does not belong to the 1-manifold. While the map from $\mathbb{R}^1$ to $\mathbb{M}$ is continuous its inverse is not.

Morse-Smale functions. The stable manifolds do not necessarily form a complex. Specifically, it is possible that the boundary of a stable manifold is not the union of other stable manifolds of lower dimension. Take for example the upright torus in Figure V.5. The stable 1-manifold of the upper saddle, w, reaches down to the lower saddle, v, but the latter is not a stable 0-manifold. The reason for this deficiency is a degeneracy in the gradient flow. In particular, we have an integral line that originates at a saddle and ends at another saddle. Equivalently, the integral line belongs to the stable 1-manifold of w and to the unstable 1-manifold of v. Generically, such integral lines do not exist.

DEFINITION. A Morse function $f : \mathbb{M} \to \mathbb{R}$ is a *Morse-Smale function* if the stable and unstable manifolds defined by the critical points of f intersect transversally.

Roughly, this requires that the stable and unstable manifolds cross when they intersect. More formally, let $\sigma : \mathbb{R}^p \to \mathbb{M}$ and $\upsilon : \mathbb{R}^q \to \mathbb{M}$ be two immersions. Letting $z \in \mathbb{M}$ be a point in their common image we say that σ and υ *intersect transversally* at z if the derived images of the tangent spaces at preimages $x \in \sigma^{-1}(z)$ and $y \in \upsilon^{-1}(z)$ span the entire tangent space of $\mathbb{M}$ at z,

$$\mathrm{D}\sigma_x(\mathrm{T}\mathbb{R}^p_x) + \mathrm{D}\upsilon_y(\mathrm{T}\mathbb{R}^q_y) \quad = \quad \mathrm{T}\mathbb{M}_z.$$

We say that σ and υ are *transversal* to each other if they intersect transversally at every point z in their common image.

Complexes. Assuming transversality, the intersection of a stable p-manifold and an unstable q-manifold has dimension $p+q-d$. Furthermore, the boundary of every stable manifold is a union of stable manifolds of lower dimension. The set of stable manifolds thus forms a complex which we construct one dimension at a time.

0**-skeleton:** add all minima as stable 0-manifolds to initialize the complex;

1**-skeleton:** add all stable 1-manifolds, each an open interval glued at its endpoints to two points in the 0-skeleton;

2**-skeleton:** add all stable 2-manifolds, each an open disk glued along its boundary circle to a cycle in the 1-skeleton;

etc. It is possible that the two minima are the same so that the interval whose ends are both glued to it forms a loop. Similarly, the cycle in the 1-skeleton can

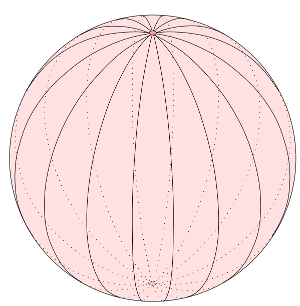

Figure V.6: All integral lines of the height function of $\mathbb{S}^2$ originate at the minimum and end at the maximum. We therefore have two stable manifolds, a vertex for the minimum and an open disk for the maximum.

be degenerate, such as pinched or even just a single point. Similar situations are possible for higher-dimensional stable manifolds. An example is the height function of the d-sphere. It has a single minimum, a single maximum, and no other critical points. The minimum has index 0 and forms a vertex in the complex. The maximum has index d and defines a stable d-manifold. It wraps around the sphere and its boundary is glued to a single point, the minimum, as illustrated for $d = 2$ in Figure V.6.

Morse inequalities. If we take the alternating sum of the stable manifolds in the above example we get $1 + (-1)^d$, which is the Euler characteristic of the d-sphere. This is not a coincidence. More generally, the alternating sum of stable manifolds gives the Euler characteristic, and this equation is part of the collection of strong Morse inequalities. We state both, the weak and the strong Morse inequalities, writing c_p for the number of critical points of index p.

MORSE INEQUALITIES. Let $\mathbb{M}$ be a manifold of dimension p and $f : \mathbb{M} \to \mathbb{R}$ a Morse function. Then

(i) WEAK: $c_p \geq \beta_p(\mathbb{M})$ for all p;
(ii) STRONG: $\sum_{p=0}^{j} (-1)^{j-p} c_p \geq \sum_{p=0}^{j} (-1)^{j-p} \beta_p(\mathbb{M})$ for all j.

As mentioned above, the strong Morse inequality for $j = d$ is an equality. We can recover the weak inequalities from the strong ones. Indeed

$$\begin{aligned} \sum_{p=0}^{j} (-1)^{j-p} c_p &\geq \beta_j(\mathbb{M}) - \sum_{p=0}^{j-1} (-1)^{j-p-1} \beta_p(\mathbb{M}) \\ &\geq \beta_j(\mathbb{M}) - \sum_{p=0}^{j-1} (-1)^{j-p-1} c_p. \end{aligned}$$

Removing the common terms on both sides leaves $c_j \geq \beta_j(\mathbb{M})$, the j-th weak inequality. We omit the proof of the strong inequalities and instead refer to the proof of their PL versions in the next section.

Floer homology. Assuming a Morse-Smale function, we can intersect the stable and unstable manifolds and get a refinement of the two complexes which we refer to as the *Morse-Smale complex* of f. Its vertices are the critical points and its cells are the components of the unions of integral lines with common origin and common destination. It is quite possible that the stable manifold of a critical point intersects the unstable manifold of another critical point in more than one component. By definition of transversality, the index difference between the origin and the destination equals the dimension of the cell. In particular, the edges are isolated integral lines connecting index p with index $p+1$ critical points.

To recover the homology of the manifold, we set up a chain complex. The p-chains are the formal sums of index p critical points. The boundary of an index p critical point u is the sum of index $p-1$ critical points connected to u by an edge in the Morse-Smale complex. If there are multiple edges, we add the index $p-1$ point multiple times. We illustrate this construction with the

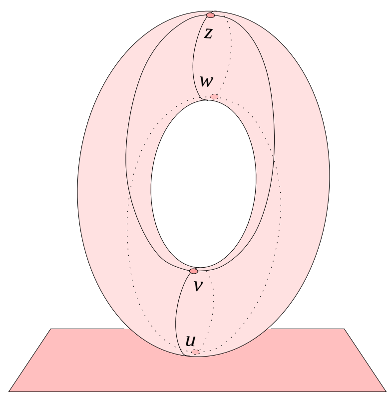

Figure V.7: The Morse-Smale complex of the height function for the almost but not entirely upright torus.

example depicted in Figure V.7. We have a slightly tilted torus whose height function is a Morse-Smale function. There are one minimum, two saddles, and one maximum. The non-trivial chain groups are therefore $\mathsf{C}_0 \simeq \mathbb{Z}_2$, $\mathsf{C}_1 \simeq \mathbb{Z}_2^2$, $\mathsf{C}_2 \simeq \mathbb{Z}_2$. The boundary of each one of the four critical points is zero. It follows that the boundary groups are trivial and the cycle groups as well as the homology groups are isomorphic to the chain groups. The Betti numbers are therefore $\beta_0 = 1$, $\beta_2 = 2$, $\beta_2 = 1$, which is consistent with what we already know about the torus.

Bibliographic notes. The concepts of integral lines and stable as well as unstable manifolds rely on fundamental properties of solutions to ordinary differential equations, in particular the Theorems of Existence and Uniqueness, see e.g. Arnold [1]. The extra requirement of transversality between stable and unstable manifolds that distinguishes Morse from Morse-Smale complexes has been proven to be generic by Kupka [3] and Smale [4]. The chain complex whose groups are formal sums of critical points is sometimes referred to as Morse-Smale-Witten complex and the resulting homology theory is referred to as Floer homology [2].

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