VI.3 An Application to Curves

In this section, we use the stability of persistence to prove an inequality that connects the length and total curvature of two curves. We begin by recasting the statement of stability in terms of continuous functions instead of filtrations.

Sublevel sets. Let $\mathbb{X}$ be a topological space and $g : \mathbb{X} \to \mathbb{R}$ a continuous function. Given a threshold $a \in \mathbb{R}$, the *sublevel set* consists of all points $x \in \mathbb{X}$ with function value less than or equal to a, $\mathbb{X}_a = g^{-1}(\infty, a]$. Similar to the complexes in a filtration, the sublevel sets are nested and give rise to sequences of homology groups connected by maps induced by inclusion, one for each dimension. Writing $f_p^{a,b} : \mathsf{H}_p(\mathbb{X}_a) \to \mathsf{H}_p(\mathbb{X}_b)$ for the map from the homology group of the sublevel set for a to that for b, we call its image a *persistent homology group*. The corresponding *persistent Betti number* is $\beta_p^{a,b} = \operatorname{rank} \operatorname{im} f_p^{a,b}$. Furthermore, $a \in \mathbb{R}$ is a *homological critical value* if there is no $\varepsilon > 0$ for which $f_p^{a-\varepsilon,a+\varepsilon}$ is an isomorphism for each dimension p. We assume that g is *tame*, by which we mean that it has only finitely many homological critical values and every sublevel set has only finite rank homology groups. Let $a_1 < a_2 < \ldots < a_n$ be the homological critical values and $b_0 < b_1 < \ldots < b_n$ interleaved values with $b_{i-1} < a_i < b_i$ for $1 \leq i \leq n$. The 1-parameter family of p-th homology groups can therefore be replaced by the finite sequence

$$0 = \mathsf{H}_p^{b_{-1}} \to \mathsf{H}_p^{b_0} \to \mathsf{H}_p^{b_1} \to \ldots \to \mathsf{H}_p^{b_n} \to \mathsf{H}_p^{b_{n+1}} = 0,$$

where $\mathsf{H}_p^b = \mathsf{H}_p(\mathbb{X}_b)$ and the groups at the two ends are added for convenience. Finally, we add $a_0 = -\infty$ and $a_{n+1} = \infty$ to the list of critical values. For $0 \leq i < j \leq n+1$, the *multiplicity* of the pair (a_i, a_j) is now defined as

$$\mu_p^{i,j} \quad = \quad (\beta_p^{b_i,b_{j-1}} - \beta_p^{b_i,b_j}) - (\beta_p^{b_{i-1},b_{j-1}} - \beta_p^{b_{i-1},b_j}).$$

To get the *dimension p persistence diagram* of g we draw each point (a_i, a_j) with multiplicity $\mu_p^{i,j}$. In contrast to the case of filtrations in which contiguous complexes differ by only one simplex, the multiplicities are no longer restricted to 0 and 1 and there can be points at infinity. In particular for a bounded function g the homology classes that get born but do not die correspond to points in the diagrams with ∞ as their second coordinate. With these definitions, we have the following stability result which we state without proof.

Stability Theorem for Functions. Let $\mathbb{X}$ be a triangulable topological space, $g, g_0 : \mathbb{X} \to \mathbb{R}$ two tame functions, and p any dimension. Then the bottleneck distance between their diagrams satisfies $d_B(\mathrm{Dgm}_p(g), \mathrm{Dgm}_p(g_0)) \leq \|g - g_0\|_\infty$.

Closed curves. We consider a closed curve $\gamma : \mathbb{S}^1 \to \mathbb{R}^2$, with or without self-intersections. Assuming γ is smooth, we have derivatives of all orders. The *speed at a point* $\gamma(s)$ is the length of the velocity vector, $\|\dot\gamma(s)\|$. We can use it to compute the length as the integral over the curve,

$$L(\gamma) \quad = \quad \int_{s \in \mathbb{S}^1} \|\dot\gamma(s)\| \, \mathrm{d}s.$$

It is convenient to assume a constant speed parametrization, that is, $\varrho = \|\dot\gamma(s)\| = \frac{1}{2\pi} L(\gamma)$ for all $s \in \mathbb{S}^1$. With this assumption, the *curvature at a point* $\gamma(s)$ is the norm of the second derivative divided by the square of the speed, $\kappa(s) = \|\ddot\gamma(s)\|/\varrho^2$. One over the curvature is the radius of the circle that best approximates the shape of the curve at the point $\gamma(s)$. To interpret this formula geometrically, we follow the velocity vector as we trace out the curve. Since its length is constant it sweeps out a circle of radius ϱ, as illustrated in Figure VI.10. The curvature is the speed at which the unit tangent vector

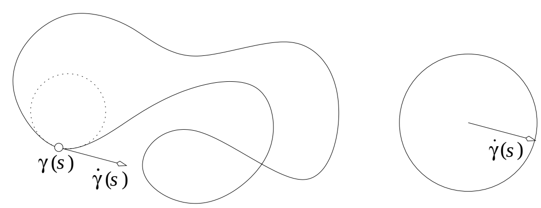

Figure VI.10: A curve with constant speed parametrization and its velocity vector sweeping out a circle with radius equal to the speed.

sweeps out the unit circle as we move the point with unit speed along the curve. This explains why we divide by the speed twice, first to compensate for the length of the velocity vector and second to compensate for the actual speed. The *total curvature* is the distance traveled by the unit tangent vector,

$$K(\gamma) \quad = \quad \int_{s \in \mathbb{S}^1} \varrho \kappa(s) \, \mathrm{d}s.$$

As an example consider the constant speed parametrization of the circle with radius r, $\gamma(s) = rs$. Writing a point of the unit circle in terms of its angle we get

$$s = \begin{bmatrix} \cos\varphi \\ \sin\varphi \end{bmatrix}, \quad \gamma(s) = \begin{bmatrix} r\cos\varphi \\ r\sin\varphi \end{bmatrix}, \quad \dot\gamma(s) = \begin{bmatrix} -r\sin\varphi \\ r\cos\varphi \end{bmatrix}.$$

The constant speed is therefore $\varrho = r$ and the length is $L(\gamma) = \int r \, \mathrm{d}s = 2\pi r$. The second derivative is $\ddot\gamma(s) = -\gamma(s)$ and the curvature is $\kappa(s) = \frac{1}{r}$ which is independent of the point on the circle. The total curvature is $K(\gamma) = \int \frac{r}{r} \, \mathrm{d}s = 2\pi$, which is independent of the radius. Indeed, the unit tangent vector travels once around the unit circle, no matter how small or how big the parametrized circle is.

Integral geometry. The length and total curvature of a curve can also be expressed in terms of integrals of elementary quantities. We begin with the length. Take a unit length line segment in the plane. The lines that cross the line segment at an angle φ form a strip of width $\sin\varphi$. Integrating over all angles gives $\int_{\varphi=0}^{\pi} \sin\varphi \, \mathrm{d}\varphi = [-\cos\varphi]_0^{\pi} = 2$. In words, the integral of the number of intersections over all lines in the plane is twice the length of the line segment. Since we can approximate the curve by a polygon whose total length approaches that of the curve, the same holds for the curve. To express this result, we write each line as the preimage of a linear function. Given a direction $u \in \mathbb{S}^1$ let $h_u : \mathbb{R}^2 \to \mathbb{R}$ be defined $h_u(x) = \langle u, x \rangle$. The line with normal direction u and offset z is $h_u^{-1}(z)$. The intersections between γ and this line corresponds to the preimage of the composition, $h^{-1}(z)$, where $h = h_u \circ \gamma$. The length of the curve is therefore as given by the Cauchy-Crofton formula,

$$L(\gamma) \quad = \quad \frac{1}{4} \int_{u \in \mathbb{S}^1} \int_{z \in \mathbb{R}} \operatorname{card}\left(h^{-1}(z)\right) \mathrm{d}z \, \mathrm{d}u.$$

To get an alternative interpretation of the total curvature, we again consider a direction $u \in \mathbb{S}^1$ and the height function in that direction, $h_u : \mathbb{S}^1 \to \mathbb{R}$ defined by $h_u(s) = \langle u, \gamma(s) \rangle$. For generic directions u, this height function has a finite number of minima and maxima, as illustrated in Figure VI.11. Recall that the

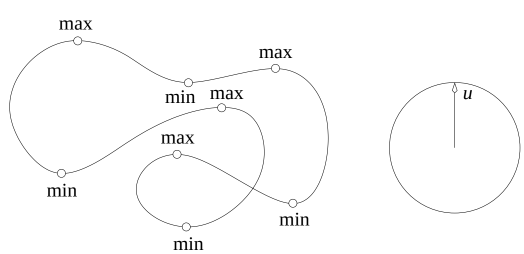

Figure VI.11: The vertical height function defined on the curve has four local minima which alternate with the four local maxima along the curve.

total curvature is the length traveled by the unit tangent vector. Equivalently, it is the length traveled by the outward unit normal vector. The number of maxima of h_u is the number of times the unit normal passes over $u \in \mathbb{S}^1$ and the number of minima is the number of times it passes over $-u \in \mathbb{S}^1$. Writing $c(h_u)$ for the number of minima and maxima of h_u, we therefore have

$$K(\gamma) = \frac{1}{2} \int_{u \in \mathbb{S}^1} c(h_u) \, \mathrm{d}u.$$

This is also π times the average number of critical points in any direction.

Theorems relating length with total curvature. Suppose the image of γ fits inside the unit disk in the plane, $\operatorname{im} \gamma \subseteq \mathbb{B}^2$. Then γ must turn to avoid crossing the boundary circle of the disk. We can therefore expect that the total curvature is bounded from below by some constant times the length. A classic result in geometry asserts that this constant is one.

FÁRY'S THEOREM. Let $\gamma : \mathbb{S}^1 \to \mathbb{R}$ be a smooth closed curve with $\operatorname{im} \gamma \subseteq \mathbb{B}^2$. Then its length is at most its total curvature, $L(\gamma) \leq K(\gamma)$.

To generalize this result we consider two curves, $\gamma, \gamma_0 : \mathbb{S}^1 \to \mathbb{R}^2$. To define the *Fréchet distance* between them we consider a homeomorphism for which we record the largest distance between corresponding points and we take the infimum over all homeomorphisms, $d_F(\gamma, \gamma_0) = \inf_\eta \max_u \|\gamma(u) - \gamma_0(\eta(u))\|$, where $\eta : \mathbb{S}^1 \to \mathbb{S}^1$ and $u \in \mathbb{S}^1$. This notion of distance does not depend on the parametrization of the two curves.

GENERALIZED FÁRY THEOREM. Let $\gamma, \gamma_0 : \mathbb{S}^1 \to \mathbb{R}^2$ be two smooth closed curves. Then $|L(\gamma) - L(\gamma_0)| \leq [K(\gamma) + K(\gamma_0) - 2\pi] \, d_F(\gamma, \gamma_0)$.

To see that Fáry's Theorem is indeed a special case, let the image of γ be contained in the unit disk and let the image of γ_0 be a tiny circle centered at the origin, as in Figure VI.12. Since γ_0 is a circle, its total curvature is 2π. Furthermore, we can make it arbitrarily small so its length approaches zero and the Fréchet distance between the two curves approaches one. Substituting 0 for $L(\gamma_0)$, 2π for $K(\gamma_0)$, and 1 for $d_F(\gamma, \gamma_0)$ gives Fáry's Theorem.

Proof in two steps. The connection between the Generalized Fáry Theorem and the stability of persistence diagrams is furnished by the integral geometry interpretations of length and total curvature. In particular, a point in the diagrams of a height function corresponds to two critical points and its persistence measures the set of normal lines that intersect the curve between these two critical points. We proceed in two steps, first considering a fixed direction

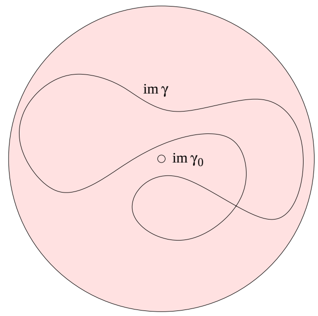

Figure VI.12: Two curves inside the unit disk. The Fréchet distance between the tiny circle and the other curve goes to one as the circle shrinks toward the origin.

and transforming the intersections counted by the Cauchy-Crofton formula into points in the persistence diagrams of the height function. Second, we integrate over all directions and this way obtain the result.

Fix a direction $u \in \mathbb{S}^1$ and let $h = h_u \circ \gamma$ and $h_0 = h_u \circ \gamma_0$ be the restrictions of the height function to the two curves. Almost all level sets of the form $h^{-1}(z)$ consist of an even number of points decomposing γ into the same number of arcs half of which belong to $h^{-1}(-\infty, z]$. Hence

$$\int_{z \in \mathbb{R}} \operatorname{card}(h^{-1}(z))\,\mathrm{d}z \quad = \quad 2\int_{z \in \mathbb{R}} \chi(z)\,\mathrm{d}z$$

for almost all z, where $\chi(z)$ is the Euler characteristic of the sublevel set defined by z. For a curve, the Euler characteristic of the sublevel set is the number of components minus the number of loops. Equivalently, $\chi(z)$ is the number of points in $\mathrm{Dgm}_0(h)$ within the quadrant $Q_z = [-\infty, z] \times (z, \infty]$ minus the number of points in $\mathrm{Dgm}_1(h)$ within the same quadrant. The same holds for γ_0 so we can compare the integrals of χ and χ_0, where $\chi_0(z)$ is the Euler characteristic of the sublevel set $h_0^{-1}(-\infty, z]$. Let $\varepsilon = d_F(\gamma, \gamma_0)$ and $\eta : \mathbb{S}^1 \to \mathbb{S}^1$ a homeomorphism such that the Euclidean distance between points $\gamma(s)$ and $\gamma_0(\eta(s))$ is at most $\varepsilon + \delta$. The stability of persistence diagrams implies that the bottleneck distance between $\mathrm{Dgm}_0(h)$ and $\mathrm{Dgm}_0(h_0)$ is at most $\varepsilon + \delta$, and the same is true for $\mathrm{Dgm}_1(h)$ and $\mathrm{Dgm}_1(h_0)$. Let ψ_0 and ψ_1 be the corresponding bijections between persistence diagrams. To bound the difference between the integrals of Euler characteristics, we observe that the contributions of the points $v \in \mathrm{Dgm}_p(h)$ and $\psi_p(v) \in \mathrm{Dgm}_p(h_0)$ cancel each other except for values of z for which one of the points lies inside the quadrant Q_z and the other lies outside. The integral of values z for which this is the case is at most $2(\varepsilon + \delta)$. In the worst case all finite points are matched with points on the diagonal in the other diagram. Points at infinity are necessarily matched with each other, and we have precisely four such points, one each in the two diagrams of h and the two diagrams of h_0. Hence, we get a contribution of $\varepsilon + \delta$ for each finite coordinate but only half that contribution for the finite coordinates of the four points at infinity. Equivalently, we get $\varepsilon + \delta$ for each minimum and each maximum of h and of h_0 except for two. More formally, $\int |\chi(z) - \chi_0(z)| \, \mathrm{d}z \leq \varepsilon[c(h) + c(h_0) - 2]$. We are now ready to integrate over all directions, plugging the bound for the Euler characteristics into the Cauchy-Crofton formula and get

$$
\begin{aligned}
|L(h) - L(h_0)| &\leq \frac{1}{4} \int_{u \in \mathbb{S}^1} \int_{z \in \mathbb{R}} |\mathrm{card}\, h^{-1}(z) - \mathrm{card}\, h_0^{-1}(z)| \, \mathrm{d}z \, \mathrm{d}u \\
&\leq \frac{1}{2} \int_{u \in \mathbb{S}^1} \int_{z \in \mathbb{R}} |\chi(z) - \chi_0(z)| \, \mathrm{d}z \, \mathrm{d}u \\
&\leq \frac{\varepsilon}{2} \int_{u \in \mathbb{S}^1} [c(h) + c(h_0) - 2] \, \mathrm{d}u.
\end{aligned}
$$

The right hand side is $\varepsilon\pi$ times the average number of minima and maxima of h plus the same for h_0 minus 2. Using the integral geometry interpretation of the total curvature, this is at most $\varepsilon[K(\gamma) + K(\gamma_0) - 2\pi]$, as claimed by the Generalized Fáry Theorem.

Bibliographic notes. The proof of the stability of persistence diagrams for functions is due to Cohen-Steiner, Edelsbrunner and Harer [2]. The proof is essentially algebraic and considerably more involved than the combinatorial proof of the version for filtrations presented in the last section. The inequality that connects the length with the total curvature of a closed curve is due to Fáry [3]. The generalization that compares the lengths of curves that are close in the Fréchet distance sense is more recent [1]. Both results have generalizations to curves in dimensions beyond two. The integral geometry interpretations of length and total curvature can be found in Santaló [4].

[1] D. Cohen-Steiner and H. Edelsbrunner. Inequalities for the curvature of curves and surfaces. *Found. Comput. Math.*, to appear.

[2] D. Cohen-Steiner, H. Edelsbrunner and J. Harer. Stability of persistence diagrams. *Discrete Comput. Geom.*, to appear.

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