CPS 196.2
Bayesian games and their use in auctions

Vincent Conitzer

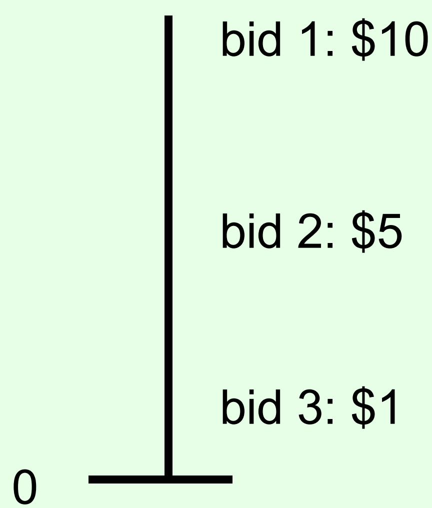
conitzer@cs.duke.edu

What is mechanism design?

- In mechanism design, we get to design the game (or mechanism)
  - e.g. the rules of the auction, marketplace, election, …
- Goal is to obtain good outcomes when agents behave strategically (game-theoretically)
- Mechanism design often considered part of game theory
- Sometimes called “inverse game theory”
  - In game theory the game is given and we have to figure out how to act
  - In mechanism design we know how we would like the agents to act and have to figure out the game
- 2007 Nobel Prize in Economics!

Example: (single-item) auctions

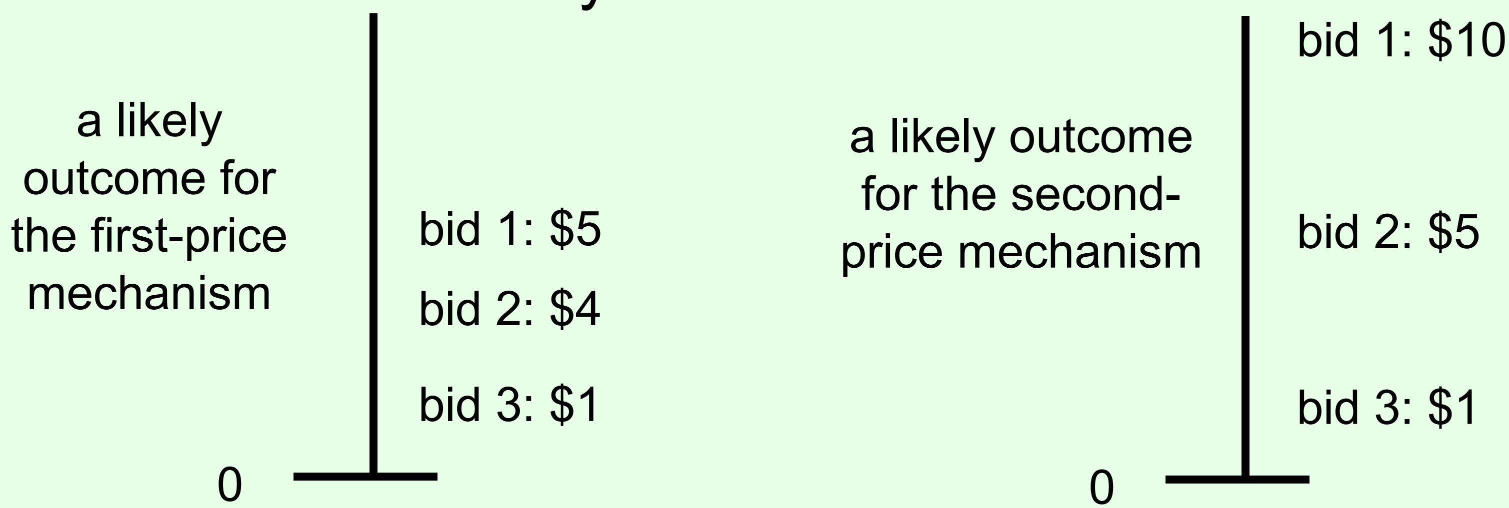
- Sealed-bid auction: every bidder submits bid in a sealed envelope
- First-price sealed-bid auction: highest bid wins, pays amount of own bid
- Second-price sealed-bid auction: highest bid wins, pays amount of second-highest bid



first-price: bid 1 wins, pays $10
second-price: bid 1 wins, pays $5

Which auction generates more revenue?

- Each bid depends on
  - bidder's true valuation for the item (utility = valuation - payment),
  - bidder's beliefs over what others will bid (→ game theory),
  - and... the auction mechanism used
- In a first-price auction, it does not make sense to bid your true valuation
  - Even if you win, your utility will be 0…
- In a second-price auction, (we will see later that) it always makes sense to bid your true valuation



Are there other auctions that perform better? How do we know when we have found the best one?

Bidding truthfully is optimal in the Vickrey auction!

- What should a bidder with value v bid?

Option 1: Win the item at price b, get utility v - b

b = highest bid among other bidders

Option 2: Lose the item, get utility 0

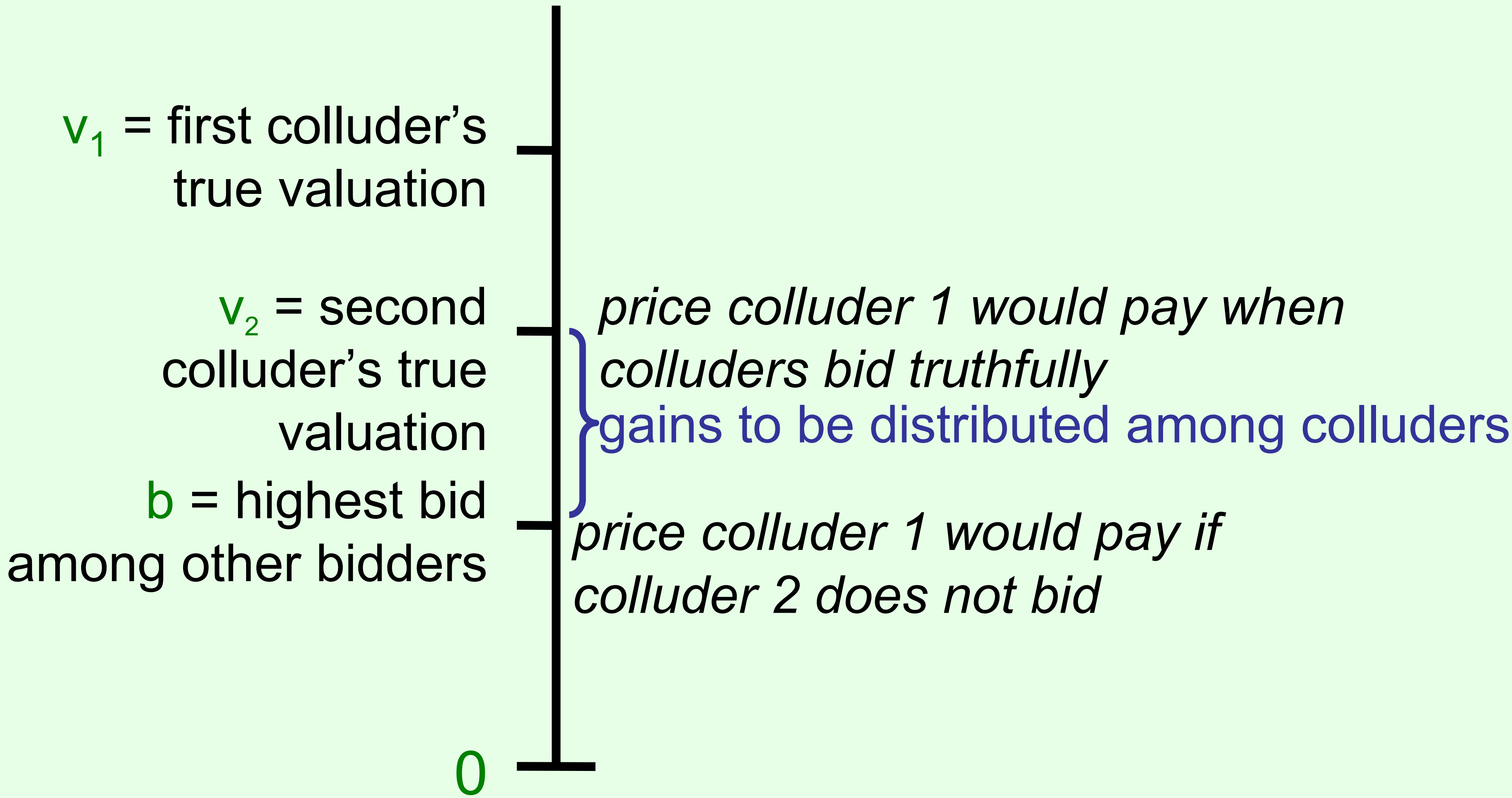
0

Would like to win if and only if v - b > 0 – but bidding truthfully accomplishes this!

We say the Vickrey auction is strategy-proof

Collusion in the Vickrey auction

- Example: two colluding bidders



Bayesian games

- In a Bayesian game a player's utility depends on that player's type as well as the actions taken in the game
  - Notation: θ_i is player i's type, drawn according to some distribution from set of types Θ_i
  - Each player knows/learns its own type, not those of the others, before choosing action
    - Pure strategy s_i is a mapping from Θ_i to A_i (where A_i is i's set of actions)
  - In general players can also receive signals about other players' utilities; we will not go into this

row player type 1 (prob. 0.5)

	L	R
U	4	6
D	2	4

column player type 1 (prob. 0.5)

	L	R
U	4	6
D	4	6

row player type 2 (prob. 0.5)

	L	R
U	2	4
D	4	2

column player type 2 (prob. 0.5)

	L	R
U	2	2
D	4	2

Converting Bayesian games to normal form

row player type 1 (prob. 0.5)

	L	R
U	4	6
D	2	4

column player type 1 (prob. 0.5)

	L	R
U	4	6
D	4	6

row player type 2 (prob. 0.5)

	L	R
U	2	4
D	4	2

column player type 2 (prob. 0.5)

	L	R
U	2	2
D	4	2

	type 1: L type 2: L	type 1: L type 2: R	type 1: R type 2: L	type 1: R type 2: R
type 1: U type 2: U	3, 3	4, 3	4, 4	5, 4
type 1: U type 2: D	4, 3.5	4, 3	4, 4.5	4, 4
type 1: D type 2: U	2, 3.5	3, 3	3, 4.5	4, 4
type 1: D type 2: D	3, 4	3, 3	3, 5	3, 4

exponential blowup in size

Bayes-Nash equilibrium

- A profile of strategies is a Bayes-Nash equilibrium if it is a Nash equilibrium for the normal form of the game
  – Minor caveat: each type should have >0 probability
- Alternative definition: for every i, for every type θ_i, for every alternative action a_i, we must have:

  $\Sigma_{\theta\text{-}i}\ P(\theta_{-i})\ u_i(\theta_i, \sigma_i(\theta_i), \sigma_{-i}(\theta_{-i})) \geq$

  $\Sigma_{\theta\text{-}i}\ P(\theta_{-i})\ u_i(\theta_i, a_i, \sigma_{-i}(\theta_{-i}))$

First-price sealed-bid auction BNE

- Suppose every bidder (independently) draws a valuation from [0, 1]
- What is a Bayes-Nash equilibrium for this?
- Say a bidder with value v_i bids $v_i(n-1)/n$
- Claim: this is an equilibrium!
- Proof: suppose all others use this strategy
- For a bid $b < (n-1)/n$, the probability of winning is $(bn/(n-1))^{n-1}$, so the expected value is $(v_i-b)(bn/(n-1))^{n-1}$
- Derivative w.r.t. b is $-(bn/(n-1))^{n-1} + (v_i-b)(n-1)b^{n-2}(n/(n-1))^{n-1}$ which should equal zero
- Implies $-b + (v_i-b)(n-1) = 0$, which solves to $b = v_i(n-1)/n$

Analyzing the expected revenue of the first-price and second-price (Vickrey) auctions

- First-price auction: probability of there not being a bid higher than b is $(bn/(n-1))^n$ (for $b < (n-1)/n$)
  - This is the cumulative density function of the highest bid
- Probability density function is the derivative, that is, it is $nb^{n-1}(n/(n-1))^n$
- Expected value of highest bid is $n(n/(n-1))^n\int^{(n-1)/n} b^n db = (n-1)/(n+1)$
- Second-price auction: probability of there not being two bids higher than b is $b^n + nb^{n-1}(1-b)$
  - This is the cumulative density function of the second-highest bid
- Probability density function is the derivative, that is, it is $nb^{n-1} + n(n-1)b^{n-2}(1-b) - nb^{n-1} = n(n-1)(b^{n-2} - b^{n-1})$
- Expected value is $(n-1) - n(n-1)/(n+1) = (n-1)/(n+1)$

Revenue equivalence theorem

- Suppose valuations for the single item are drawn i.i.d. from a continuous distribution over [L, H] (with no “gaps”), and agents are risk-neutral
- Then, any two auction mechanisms that
  - in equilibrium always allocate the item to the bidder with the highest valuation, and
  - give an agent with valuation L an expected utility of 0,

  will lead to the same expected revenue for the auctioneer

(As an aside) what if bidders are not risk-neutral?

- Behavior in second-price/English/Japanese does not change, but behavior in first-price/Dutch does
- Risk averse: first price/Dutch will get higher expected revenue than second price/Japanese/English
- Risk seeking: second price/Japanese/English will get higher expected revenue than first price/Dutch

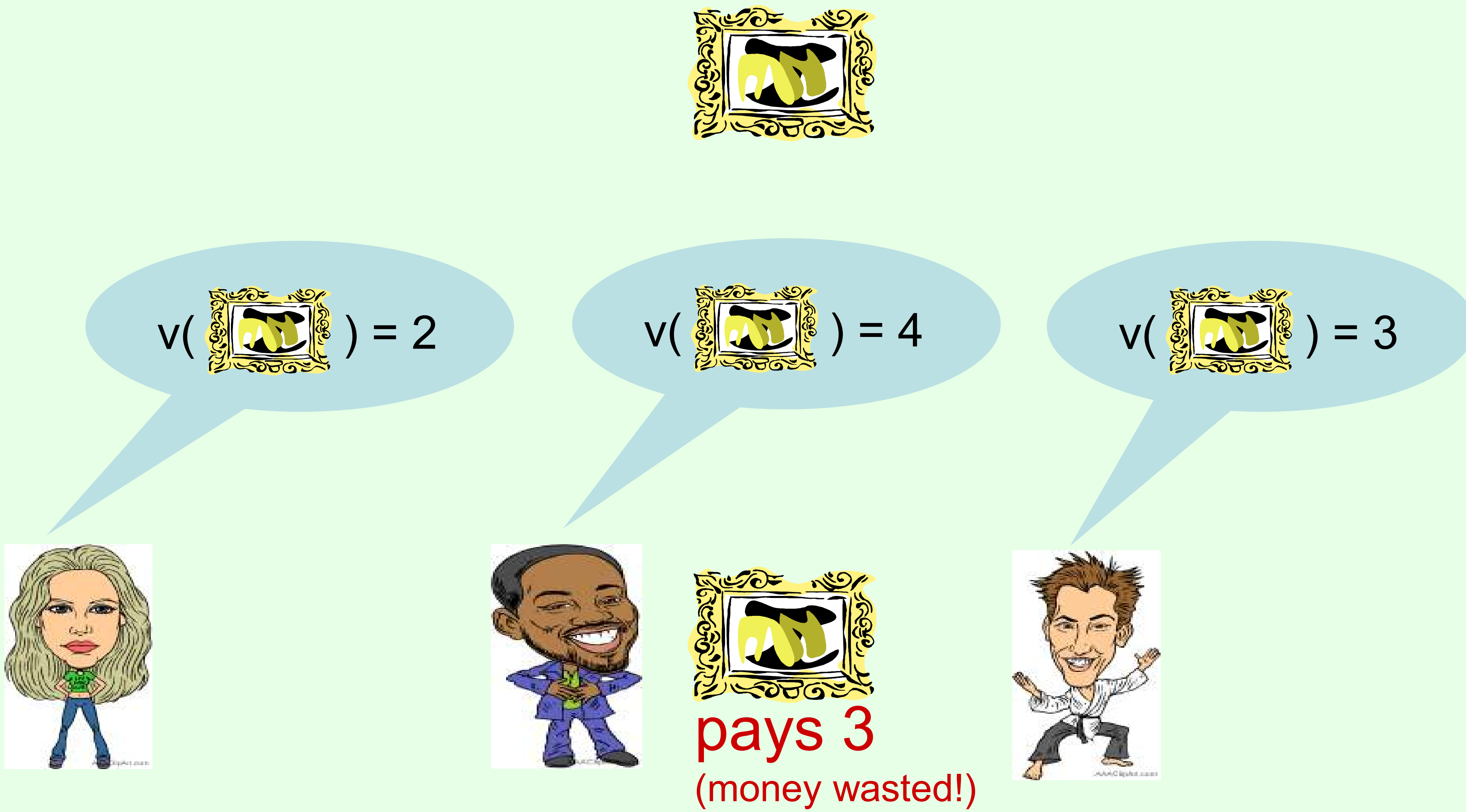
(As an aside) interdependent valuations

- E.g. bidding on drilling rights for an oil field
- Each bidder i has its own geologists who do tests, based on which the bidder assesses an expected value v_i of the field
- If you win, it is probably because the other bidders' geologists' tests turned out worse, and the oil field is not actually worth as much as you thought
  - The so-called winner's curse
- Hence, bidding v_i is no longer a dominant strategy in the second-price auction
- In English and Japanese auctions, you can update your valuation based on other agents' bids, so no longer equivalent to second-price
- In these settings, English (or Japanese) > second-price > first-price/Dutch in terms of revenue

Expected-revenue maximizing ("optimal") auctions [Myerson 81]

- Vickrey auction does not maximize expected revenue
  - E.g. with only one bidder, better off making a take-it-or-leave-it offer (or equivalently setting a reserve price)
- Suppose agent i draws valuation from probability density function f_i (cumulative density F_i)
- Bidder's virtual valuation $\psi(v_i)= v_i - (1 - F_i(v_i))/f_i(v_i)$
  - Under certain conditions, this is increasing; assume this
- The bidder with the highest virtual valuation (according to his reported valuation) wins (unless all virtual valuations are below 0, in which case nobody wins)
- Winner pays value of lowest bid that would have made him win
- E.g. if all bidders draw uniformly from [0, 1], Myerson auction = second-price auction with reserve price ½

Vickrey auction without a seller



Can we redistribute the payment?

Idea: give everyone 1/n
of the payment

not strategy-proof

Bidding higher can increase your redistribution payment

Incentive compatible redistribution

[Bailey 97, Porter et al. 04, Cavallo 06]

Idea: give everyone 1/n of second-highest **other** bid

v() = 2

v() = 4

v() = 3

receives 1

pays 3

receives 2/3

receives 2/3

2/3 wasted (22%)

Strategy-proof

Your redistribution does not depend on your bid;

incentives are the same as in Vickrey

Bailey-Cavallo mechanism…

- Bids: $V_1 \geq V_2 \geq V_3 \geq \ldots \geq V_n \geq 0$
- First run Vickrey auction
- Payment is V_2
- First two bidders receive V_3/n
- Remaining bidders receive V_2/n
- Total redistributed: $2V_3/n+(n-2)V_2/n$

$R_1 = V_3/n$
$R_2 = V_3/n$
$R_3 = V_2/n$
$R_4 = V_2/n$
...
$R_{n-1} = V_2/n$
$R_n = V_2/n$

Is this the best possible?

Another redistribution mechanism

- Bids: $V_1 \geq V_2 \geq V_3 \geq V_4 \geq \ldots \geq V_n \geq 0$
- First run Vickrey
- Redistribution:

  Receive 1/(n-2) * second-highest **other** bid,

  - 2/[(n-2)(n-3)] third-highest **other** bid
- Total redistributed:

  $V_2 - 6V_4/[(n-2)(n-3)]$

$R_1 = V_3/(n-2) - 2/[(n-2)(n-3)]V_4$

$R_2 = V_3/(n-2) - 2/[(n-2)(n-3)]V_4$

$R_3 = V_2/(n-2) - 2/[(n-2)(n-3)]V_4$

$R_4 = V_2/(n-2) - 2/[(n-2)(n-3)]V_3$

...

$R_{n-1} = V_2/(n-2) - 2/[(n-2)(n-3)]V_3$

$R_n = V_2/(n-2) - 2/[(n-2)(n-3)]V_3$

Idea pursued further in Guo & Conitzer 07 / Moulin 07