Fifth Homework Assignment

Write the solution to each problem on a single page. The deadline for handing in solutions is November 13.

Problem 1. (20 points). Let $G = (V, E)$ be a maximally connected planar graph. An *edge* 3-*coloring* is a mapping $\eta : E \to \{1, 2, 3\}$ such that $\eta(e) \neq \eta(f)$ whenever $e \neq f$ bound a common triangle. Prove that if G has a vertex 4-coloring then it also has an edge 3-coloring.

Problem 2. ($20 = 10 + 10$ points). Let K be a set of triangles together with their edges and vertices.

(a) Give an algorithm that decides whether or not K is a triangulation of a 2-manifold.

(b) Analyze your algorithm and collect credit points if the running time of your algorithm is linear in the number of triangles.

Problem 3. ($20 = 5{+}7{+}8$ points). Determine the type of 2-manifold with boundary obtained by the following constructions.

(a) Remove a cylinder from a torus in such a way that the rest of the torus remains connected.

(b) Remove a disk from the projective plane.

(c) Remove a Möbius strip from a Klein bottle.

Problem 4. ($20 = 5 + 5 + 5 + 5$ points). Recall that the sphere is the space of points at unit distance from the origin in three dimensions, $\mathbb{S}^2 = \{x \in \mathbb{R}^3 \mid \|x\| = 1\}$.

(a) Give a triangulation of $\mathbb{S}^2$.

(b) Give the corresponding boundary matrices.

(c) Reduce the boundary matrices.

(d) Give the Betti numbers of $\mathbb{S}^2$.

Problem 5. ($20 = 10 + 10$ points). The *dunce cap* is obtained by gluing the three edges of a triangular sheet of paper to each other. [After gluing the first two edges you get a cone, with the glued edges forming a seam connecting the cone point with the rim. In the final step, wrap the seam around the rim, gluing all three edges to each other. To imagine how this work, it might help to think of the final result as similar to the shell of a snale.]

(a) Is the dunce cap a 2-manifold? Justify your answer.

(b) Give a triangulation of the dunce cap, making sure that no two edges connect the same two vertices and no two triangles connect the same three vertices.