17 Geometric Graphs

In the abstract notion of a graph, an edge is merely a pair of vertices. The geometric (or topological) notion of a graph is closer to our intuition in which we think of an edge as a curve that connects two vertices.

Embeddings. Let $G = (V, E)$ be a simple, undirected graph and write $\mathbb{R}^2$ for the two-dimensional real plane. A *drawing* maps every vertex $v \in V$ to a point $\varepsilon(v)$ in $\mathbb{R}^2$, and it maps every edge $\{u, v\} \in E$ to a curve with endpoints $\varepsilon(u)$ and $\varepsilon(v)$. The drawing is an *embedding* if

1. different vertices map to different points;
2. the curves have no self-intersections;
3. the only points of a curve that are images of vertices are its endpoints;
4. two curves intersect at most in their endpoints.

We can always map the vertices to points and the edges to curves in $\mathbb{R}^3$ so they form an embedding. On the other hand, not every graph has an embedding in $\mathbb{R}^2$. The graph G is *planar* if it has an embedding in $\mathbb{R}^2$. As illustrated in Figure 73, a planar graph has many drawings, not all of which are embeddings. A *straight-line* drawing or embedding is one in which each edge is mapped to a straight line segment. It is uniquely determined by the mapping of the vertices, $\varepsilon : V \to \mathbb{R}^2$. We will see later that every planar graph has a straight-line embedding.

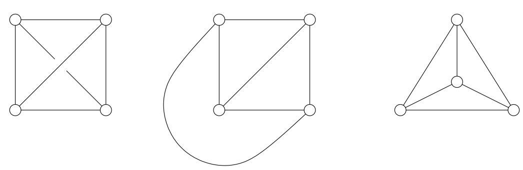

Figure 73: Three drawings of K_4, the complete graph with four vertices. From left to right: a drawing that is not an embedding, an embedding with one curved edge, a straight-line embedding.

Euler's formula. A *face* of an embedding ε of G is a component of the thus defined decomposition of $\mathbb{R}^2$. We write $n = |V|$, $m = |E|$, and ℓ for the number of faces. Euler's formula says these numbers satisfy a linear relation.

EULER'S FORMULA. If G is connected and ε is an embedding of G in $\mathbb{R}^2$ then $n - m + \ell = 2$.

PROOF. Choose a spanning tree (V, T) of $G = (V, E)$. It has n vertices, $|T| = n - 1$ edges, and one (unbounded) face. We have $n - (n - 1) + 1 = 2$, which proves the formula if G is a tree. Otherwise, draw the remaining edges, one at a time. Each edge decomposes one face into two. The number of vertices does not change, m increases by one, and ℓ increases by one. Since the graph satisfies the linear relation before drawing the edge, it satisfies the relation also after drawing the edge. ◻

A planar graph is *maximally connected* if adding any one new edge violates planarity. Not surprisingly, a planar graph of three or more vertices is maximally connected iff every face in an embedding is bounded by three edges. Indeed, suppose there is a face bounded by four or more edges. Then we can find two vertices in its boundary that are not yet connected and we can connect them by drawing a curve that passes through the face; see Figure 74. For obvious reasons, we call an embedding of a maximally connected planar graph with $n \geq 3$ vertices a *triangulation*. For such graphs, we have an additional linear relation, namely $3\ell = 2m$. We can thus rewrite Euler's formula and get $n - m + \frac{2m}{3} = 2$ and $n - \frac{3\ell}{2} + \ell = 2$ and therefore

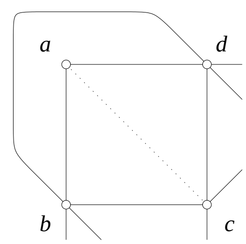

Figure 74: Drawing the edge from a to c decomposes the quadrangle into two triangles. Note that we cannot draw the edge from b to d since it already exists outside the quadrangle.

$$\begin{aligned} m &= 3n - 6; \\ \ell &= 2n - 4, \end{aligned}$$

Every planar graph can be completed to a maximally connected planar graph. For $n \geq 3$ this implies that the planar graph has at most $3n - 6$ edges and at most $2n - 4$ faces.

Forbidden subgraphs. We can use Euler's relation to prove that the complete graph of five vertices is not planar. It has $n = 5$ vertices and $m = 10$ edges, contradicting the upper bound of at most $3n - 6 = 9$ edges. Indeed, every drawing of K_5 has at least two edges crossing; see Figure 75. Similarly, we can prove that the complete bipartite

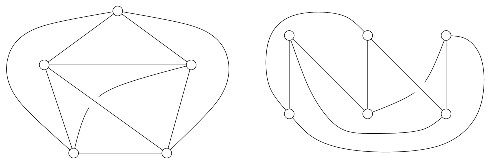

Figure 75: A drawing of K_5 on the left and of $K_{3,3}$ on the right.

graph with three plus three vertices is not planar. It has $n = 6$ vertices and $m = 9$ edges. Every cycle in a bipartite graph has an even number of edges. Hence, $4\ell \leq 2m$. Plugging this into Euler's formula, we get $n - m + \frac{m}{2} \geq 2$ and therefore $m \leq 2n - 4 = 8$, again a contradiction.

In a sense, K_5 and $K_{3,3}$ are the quintessential non-planar graphs. To make this concrete, we still need an operation that creates or removes degree-2 vertices. Two graphs are *homeomorphic* if one can be obtained from the other by a sequence of operations, each deleting a degree-2 vertex and replacing its two edges by the one that connects its two neighbors, or the other way round.

KURATOWSKI'S THEOREM. A graph G is planar iff no subgraph of G is homeomorphic to K_5 or to $K_{3,3}$.

The proof of this result is a bit lengthy and omitted.

Pentagons are star-convex. Euler's formula can also be used to show that every planar graph has a straight-line embedding. Note that the sum of vertex degrees counts each edge twice, that is, $\sum_{v \in V} \deg(v) = 2m$. For planar graphs, twice the number of edges is less than $6n$ which implies that the average degree is less than six. It follows that every planar graph has at least one vertex of degree 5 or less. This can be strengthened by saying that every planar graph with $n \geq 4$ vertices has at least four vertices of degree at most 5 each. To see this, assume the planar graph is maximally connected and note that every vertex has degree at least 3. The deficiency from degree 6 is thus at most 3. The total deficiency is $6n - \sum_{v \in V} \deg(v) = 12$ which implies that we have at least four vertices with positive deficiency.

We need a little bit of geometry to prepare the construction of a straight-line embedding. A region $R \subseteq \mathbb{R}^2$ is *convex* if $x, y \in R$ implies that the entire line segment connecting x and y is contained in R. Figure 76 shows regions of either kind. We call R *star-convex* of there is a point $z \in R$ such that for every point $x \in R$ the line segment connecting x with z is contained in R. The set of

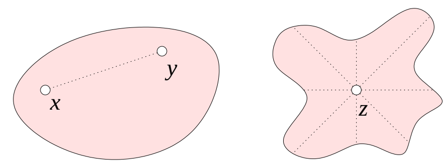

Figure 76: A convex region on the left and a non-convex star-convex region on the right.

such points z is the *kernel* of R. Clearly, every convex region is star-convex but not every star-convex region is convex. Similarly, there are regions that are not star-convex, even rather simple ones such as the hexagon in Figure 77. However, every pentagon is star-convex. Indeed, the pen-

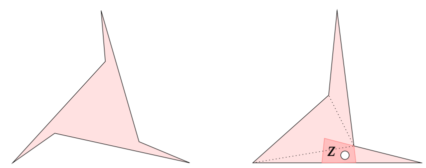

Figure 77: A non-star-convex hexagon on the left and a star-convex pentagon on the right. The dark region inside the pentagon is its kernel.

tagon can be decomposed into three triangles by drawing two diagonals that share an endpoint. Extending the incident sides into the pentagon gives locally the boundary of the kernel. It follows that the kernel is non-empty and has interior points.

Fáry's construction. We construct a straight-line embedding of a planar graph $G = (V, E)$ assuming G is maximally connected. Choose three vertices, a, b, c, connected by three edges to form the outer triangle. If G has only $n = 3$ vertices we are done. Else it has at least one vertex $u \in V = \{a, b, c\}$ with $\deg(u) \leq 5$.

`Step 1.` Remove u together with the $k = \deg(u)$ edges incident to u. Add $k - 3$ edges to make the graph maximally connected again.

`Step 2.` Recursively construct a straight-line embedding of the smaller graph.

`Step 3.` Remove the added $k - 3$ edges and map u to a point $\varepsilon(u)$ in the interior of the kernel of the resulting k-gon. Connect $\varepsilon(u)$ with line segments to the vertices of the k-gon.

Figure 78 illustrates the recursive construction. It is straightforward to implement but there are numerical issues in the choice of $\varepsilon(u)$ that limit the usefulness of this construction.

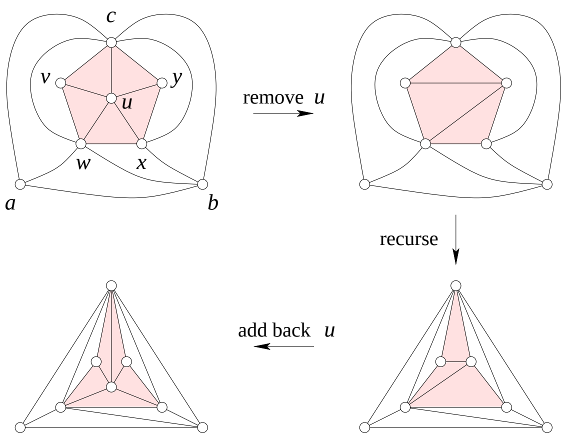

Figure 78: We fix the outer triangle, remove the degree-5 vertex, recursively construct a straight-line embedding of the rest, and finally add the vertex back.

Tutte's construction. A more useful construction of a straight-line embedding goes back to the work of Tutte. We begin with a definition. Given a finite set of points, $x_1, x_2, \ldots, x_j$, the *average* is

$$x = \frac{1}{n}\sum_{i=1}^{j} x_i.$$

For $j = 2$, it is the midpoint of the edge and for $j = 3$, it is the centroid of the triangle. In general, the average is a point somewhere between the x_i. Let $G = (V, E)$ be a maximally connected planar graph and a, b, c three vertices connected by three edges. We now follow Tutte's construction to get a mapping $\varepsilon : V \to \mathbb{R}^2$ so that the straight-line drawing of G is a straight-line embedding.

`Step 1.` Map a, b, c to points $\varepsilon(a), \varepsilon(b), \varepsilon(c)$ spanning a triangle in $\mathbb{R}^2$.

`Step 2.` For each vertex $u \in V - \{a, b, c\}$, let N_u be the set of neighbors of u. Map u to the average of the images of its neighbors, that is,

$$\varepsilon(u) = \frac{1}{|N_u|}\sum_{v \in N_u} \varepsilon(v).$$

The fact that the resulting mapping $\varepsilon : V \to \mathbb{R}^2$ gives a straight-line embedding of G is known as Tutte's Theorem. It holds even if G is not quite maximally connected and if the points are not quite the averages of their neighbors. The proof is a bit involved and omitted.

The points $\varepsilon(u)$ can be computed by solving a system of linear equations. We illustrate this for the graph in Figure 78. We set $\varepsilon(a) = \binom{-1}{-1}$, $\varepsilon(b) = \binom{1}{-1}$, $\varepsilon(c) = \binom{0}{1}$. The other five points are computed by solving the system of linear equations $\mathbf{A}\mathbf{v} = 0$, where

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 & -5 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & -3 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & -6 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & -5 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & -3 \end{bmatrix}$$

and $\mathbf{v}$ is the column vector of points $\varepsilon(a)$ to $\varepsilon(y)$. There are really two linear systems, one for the horizontal and the other for the vertical coordinates. In each system, we have $n-3$ equations and a total of $n-3$ unknowns. This gives a unique solution provided the equations are linearly independent. Proving that they are is part of the proof of Tutte's Theorem. Solving the linear equations is a numerical problem that is studies in detail in courses on numerical analysis.