CPS230 **Prof. Jeff Vitter**
Design and Analysis of Algorithms **Fall 2001**

Trial and Error Method for Recurrences

This handout discusses the trial and error method for determining the leading-order term of $S(n)$ when $S(n)$ is defined by a recurrence relation where $S(n)$ appears by itself on the lefthand side of the recurrence. It is important to note that this method is not foolproof, and when it works not even Justin Wilson can *gahhhhrawteeeeee!* that you have the right answer. (Justin Wilson is unfortunately no longer with us.) But it is very easy to use, and often gives you good results. To *prove* that you have the right answer, always resort to induction. (But once you an idea of what the answer is, the induction proof is usually straightforward.) Here's the method:

(i) Derive a recurrence relation for $S(n)$, in case it's in some other form, like summation form.
(ii) Guess the leading-order term.
(iii) Plug the leading-order term into both sides of the recurrence relation. The lefthand side will now contain nothing but the leading order term.
(iv) Compare the two sides of the equation:
   (a) If the lefthand side (*LHS*) leading-order term matches the righthand side (*RHS*) leading-order term and the second-order term on the *RHS* cancels out, your guess is correct.
   (b) Else if the *LHS* leading-order term and the *RHS* leading-order term match, then attempt to find values for the undetermined quantities, if any, that will allow you to cancel the *RHS* second-order term. If you are successful, your guess is correct.
   (c) Else if the *LHS* leading-order term is smaller than the *RHS* leading-order term, then guess another leading-order term that grows *more quickly* than your previous guess, and go to step (iii).
   (d) Else if the *LHS* leading-order term is larger than the *RHS* leading-order term, then guess another leading-order term that grows *less quickly* than your previous guess, and go to step (iii).
   (e) If all of the above fail, fudge or punt.

Example 1. $S(n) = \sum_{1 \le k \le n} k^2$.

(i) $S(n) = S(n-1) + n^2$.
(ii) Guess $S(n) \sim an^2$, where a is an undetermined constant.
(iii)

$$\begin{aligned} an^2 &\stackrel{?}{=} a(n-1)^2 + n^2 \\ &\stackrel{?}{=} (a+1)n^2 - 2an + a \end{aligned}$$

(iv) (c) The LHS leading-order term an^2 is smaller than the RHS leading-order term $(a+1)n^2$, regardless of what a is. The reason is that the difference between the guesses an^2 and $a(n-1)^2$ for $S(n)$ and $S(n-1)$ is too small. Let's make a faster-growing guess. Let's also try to be more general and not pin ourselves down. We'll try the more general guess $S(n) \sim an^c$, where a and c are undetermined constants.

(iii)

$$
\begin{aligned}
an^c &\stackrel{?}{=} a(n-1)^c + n^2 \\
&\stackrel{?}{=} a\left(n^c - \binom{c}{1}n^{c-1} + \binom{c}{2}n^{c-2} - \binom{c}{3}n^{c-3} + \ldots + (-1)^c\binom{c}{c}n^{c-c}\right) + n^2 \\
&\stackrel{?}{=} an^c - acn^{c-1} + ac(c-1)\frac{1}{2}n^{c-2} - ac(c-1)(c-2)\frac{1}{6}n^{c-3} + \ldots + a(-1)^c + n^2.
\end{aligned}
$$

(iv) The leading-order terms an^c match. Let's try (iv)(b). The only way that the *RHS* second-order term can possibly cancel out is if $-acn^{c-1} = n^2$, because all other terms (except for the leading-order term an^c) are definitely of a lower order than $-acn^{c-1}$. So, let's see if we can make n^2 cancel $-acn^{c-1}$: $-acn^{c-1} = n^2$ implies $c = 3$ which further implies $a = \frac{1}{3}$, giving us $S(N) \sim \frac{1}{3}n^3$ by substitution for c and a.

Note: the exact value of $S(n)$ is $\frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$. If you can't guess the exact answer, you can prove that $\frac{1}{3}n^3$ is the leading term by induction. To do that, for example, we could prove that $\frac{1}{3}n^3 \le S(n) \le \frac{1}{3}n^3 + \frac{2}{3}n^2$. From this it would then follow that $\lim_{n\to\infty} S(n)/(\frac{1}{3}n^3) = 1$, which means by definition that $S(n) \sim \frac{1}{3}n^3$. (To prove that the above limit is 1, use either the "squeeze theorem" from first-year calculus, or use l'Hospital's rule, which is given at the end of this handout.)

Here's how to prove that $\frac{1}{3}n^3 \le S(n) \le \frac{1}{3}n^3 + \frac{2}{3}n^2$ by induction. First we'll prove the first inequality $\frac{1}{3}n^3 \le S(n)$:

Base case. $\frac{1}{3}(1)^3 \le S(1)$.

Induction step. Assume that the inequality is true for 1, 2, ..., $n-1$, and let's prove it for n:

$$
\begin{aligned}
S(n) &= n^2 + S(n-1) \\
&\ge n^2 + \frac{1}{3}(n-1)^3 = n^2 + \frac{1}{3}n^3 - n^2 + n - \frac{1}{3} \\
&\ge \frac{1}{3}n^3.
\end{aligned}
$$

Now we'll prove $S(n) \le \frac{1}{3}n^3 + \frac{2}{3}n^2$:

Base case. $S(1) \le \frac{1}{3}(1)^3 + \frac{2}{3}(1)^2$.

Induction step.

$$
\begin{aligned}
S(n) &= n^2 + S(n-1) \\
&\le n^2 + \frac{1}{3}(n-1)^3 + \frac{2}{3}(n-1)^2 = n^2 + \frac{1}{3}n^3 - n^2 + n - \frac{1}{3} + \frac{2}{3}n^2 - \frac{4}{3}n + \frac{2}{3} \\
&= \frac{1}{3}n^3 + \frac{2}{3}n^2 - \frac{1}{3}n + \frac{1}{3} \\
&\le \frac{1}{3}n^3 + \frac{2}{3}n^2.
\end{aligned}
$$

The way we chose the particular inequality $\frac{1}{3}n^3 \le S(n) \le \frac{1}{3}n^3 + \frac{2}{3}n^2$ that we did was by experiment. Before writing this handout, we fiddled with possible second-order terms until we found some that made the above induction proof work. The terms $\frac{1}{3}n^3 + 0n^2$ worked for the lefthand side of the inequality, and $\frac{1}{3}n^3 + \frac{2}{3}n^2$ worked for the righthand side. Some other choices would work, too. For example, if you were smart (or lucky) enough to guess the closed-form solution to $S(n)$, you could prove that by induction.

Example 2. This example will be covered in a couple weeks.

$$F(n) = \begin{cases} F(n-1) + F(n-2) & \text{if } n \geq 2; \\ 1 & \text{if } n = 1; \\ 0 & \text{if } n = 0. \end{cases}$$

(i) Given.
(ii) Guess $F(n) \sim an^c$.
(iii)

$$\begin{aligned} an^c &\stackrel{?}{=} a(n-1)^c + a(n-2)^c \\ &\stackrel{?}{=} an^c - acn^{c-1} + \ldots + an^c - 2acn^{c-1} + \ldots \\ &\stackrel{?}{=} 2an^c + \ldots \end{aligned}$$

(iv) (c) The *LHS* leading-order term is too small. The function should grow more quickly in order to make the *LHS* leading-order term grow faster relative to the *RHS* leading-order term. Guess $F(n) \sim ab^n$.
(iii) $ab^n \stackrel{?}{=} ab^{n-1} + ab^{n-2} \stackrel{?}{=} ab^{n-2}(b+1)$.
(iv) The *LHS* leading-order term and *RHS* leading-order term would match if $b+1 = b^2$, so try (iv)(b): If $b^2 = b+1$, then $b^2 - b - 1 = 0$. Therefore, by the quadratic equation,

$$b = \frac{1 \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)} = \frac{1 \pm \sqrt{5}}{2}.$$

It turns out that the exact value of $F(n)$ is a linear combination of our two possibilities:

$$a_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + a_2 \left(\frac{1-\sqrt{5}}{2}\right)^n.$$

We can solve for a_1 and a_2 using the values $F(0) = 0$ and $F(1) = 1$:

$$0 = F(0) = a_1 \left(\frac{1+\sqrt{5}}{2}\right)^0 + a_2 \left(\frac{1-\sqrt{5}}{2}\right)^0 = a_1 + a_2 \qquad \Longrightarrow \qquad a_1 = -a_2.$$

$$1 = F(1) = a_1 \left(\frac{1+\sqrt{5}}{2}\right)^1 a_2 \left(\frac{1-\sqrt{5}}{2}\right)^1 = a_1 \left(\frac{1+\sqrt{5}}{2}\right) - a_1 \left(\frac{1-\sqrt{5}}{2}\right)$$

$$\Longrightarrow \qquad a_1 = \frac{1}{\sqrt{5}} \text{ and } a_2 = -\frac{1}{\sqrt{5}}.$$

Therefore,

$$F(n) = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n \right),$$

which can be proved rigorously by induction. In particular, we have

$$F(n) \sim \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n, \quad \text{since } \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^n \to 0 \text{ as } n \to \infty.$$

Example 3. This example will be discussed when we cover asymptotics near the end of the semester.

$$P(n) = n! = \begin{cases} nP(n-1) & \text{if } n \geq 1; \\ 1 & \text{if } n = 0. \end{cases}$$

(i) Given.
(ii) Guess $P(n) \sim n^c$.
(iii) $n^c \stackrel{?}{=} n(n-1)^c \stackrel{?}{=} n(n^c + O(n^{c-1})) \stackrel{?}{=} n^{c+1} + O(n^c)$.
(iv) (c) The *RHS* is too big for the guess to work. The reason is that the difference between the guesses n^c and n^{c-1} for $P(n)$ and $P(n-1)$ is too small. Therefore, $P(n)$ should grow faster. Guess $P(n) \sim a^n$.
(iii) $a^n \stackrel{?}{=} na^{n-1} \stackrel{?}{=} (n/a)a^n$.
(iv) (c) Again, the *RHS* is too big for the guess to work. Therefore, $P(n)$ should grow faster. Guess $P(n) \sim n^n$.
(iii) $n^n \stackrel{?}{=} n(n-1)^{n-1} \stackrel{?}{=} n^n(1-1/n)^{n-1} \sim n^n(1/e)$, by L'Hospital's rule.
(iv) (d) Now, the *RHS* is too small. Hence, $P(n)$ should grow slower. Guess $P(n) \sim b(n/a)^{n+c}$.
(iii)

$$b\left(\frac{n}{a}\right)^{n+c} \stackrel{?}{=} nb\left(\frac{(n-1)}{a}\right)^{n-1+c} \stackrel{?}{=} b\left(\frac{n}{a}\right)^{n+c} a\left(1-\frac{1}{n}\right)^{n+1-c}. \qquad (*)$$

(iv) (b) L'Hospital's Rule says that $\lim_{n\to\infty}(1-1/n)^{n+1-c} = 1/e$. But we must make sure that the second-order term in the expansion of the *RHS* is 0. Let's expand:

$$\begin{aligned}
\left(1-\frac{1}{n}\right)^{n+1-c} &= \exp\left(\ln\left(1-\frac{1}{n}\right)^{n+1-c}\right) \\
&= \exp\left((n+1+c)\ln\left(1-\frac{1}{n}\right)\right) \\
&= \exp\left((n+1-c)\left(-\frac{1}{n}-\frac{1}{2n^2}+O\left(\frac{1}{n^3}\right)\right)\right) \\
&= \exp\left(\left(-\frac{n+1-c}{n}-\frac{n+1-c}{2n^2}+O\left(\frac{1}{n^2}\right)\right)\right) \\
&= \exp\left(\left(-1+\frac{1-c}{n}-\frac{1}{2n}+\frac{1-c}{2n^2}+O\left(\frac{1}{n^2}\right)\right)\right) \\
&= e^{-1}\exp\left(\left(\frac{\frac{1}{2}-c}{n}+O\left(\frac{1}{n^2}\right)\right)\right) \\
&= e^{-1}\left(1+\left(\frac{1}{2}-c\right)\frac{1}{n}+O\left(\frac{1}{n^2}\right)\right). \qquad (**)
\end{aligned}$$

In order for the leading-order term of the *RHS* of $(*)$ to equal the *LHS* of $(*)$ and for the second-order term of the *RHS* to be 0, we must have

$$a\left(1-\frac{1}{n}\right)^{n+1-c} = 1 + O\left(\frac{1}{n^2}\right).$$

Substituting $(**)$, we get

$$ae^{-1}\left(1+\left(\frac{1}{2}-c\right)\frac{1}{n}+O\left(\frac{1}{n^2}\right)\right)=1+O\left(\frac{1}{n^2}\right) \qquad \Longrightarrow \qquad a=e \text{ and } c=\frac{1}{2}.$$

Note that b has not been determined. This suggests that $P(n)\sim b(n/e)^{n+\frac{1}{2}}$ for some unknown constant b, or in other words, $P(n)=\Theta(n/e)^{n+\frac{1}{2}}$. That is the best this method can do.
It turns out (by Stirling's approximation) that the actual value of b is $\sqrt{2\pi e}$, so

$$n!\sim\sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$$

Note that in the material that follows, $\log n$ and $\lg n$ both denote the natural logarithm of n.