The Push Relabel Algorithm (Goldberg – Tarjan)

FF approach: Maintain valid flow and keep increasing until t not reachable from s in residual network

Push Relabel approach: Maintain t not reachable from s in residual network and keep changing flow until it becomes valid.

Preflow: $f: (x,y) \to \mathbb{R}_+$

(1) $f(x,y) \le u(x,y)$

(2) $\forall y \in V \setminus \{t\}$, $\underbrace{\sum_x f(x,y) - \sum_z f(y,z)}_{\text{excess flow at } y} \ge 0$

A preflow where excess flow at all vertices $y \in V \setminus \{s,t\}$ is 0 is a valid flow.

Residual network for a preflow defined as before:

$$u_f(x,y) = u(x,y) - f(x,y)$$
$$u_f(y,x) = u(y,x) + f(x,y)$$

Push: If excess at $y \ge r$, and $u_f(y,w) \ge r$, send preflow of r on edge (y,w) in residual network: compose preflow on residual network with preflow in original graph as for flows.

ALGORITHM

Initial.:
- Saturate all edges out of s with preflow
- $h(s) = |V|$

Repeat until preflow is feasible
- Push excess flow on edge along decreasing height
- Relabel vertex with excess flow by incrementing height by 1

Output feasible flow

Lemma 1: For every vertex x, $h(x) \le 2|V| - 1$
($\Rightarrow$ # of relabel operations $\le 2|V|^2$)

Lemma 2: # of saturating pushes on an edge $\le |V|$
($\Rightarrow$ # of saturating pushes $\le 2|E||V|$)

Lemma 3: # of non-saturating pushes $\le 4|V|^2|E|$

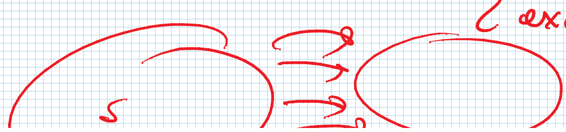
Lemma 4: All push/relabel operations take $O(1)$ time

Theorem: Time complexity is $O(|V|^2|E|)$.

Proof of Lemma 1

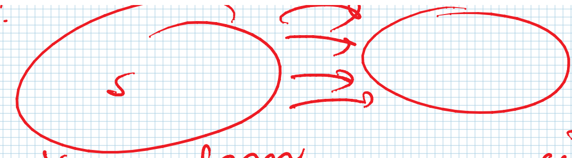
Lemma: If a vertex y has excess flow > 0, then $\exists$ $v-s$ path in residual network.

Proof:



$\sum$ excess flow
≥ 0 by defn
≤ 0 because internal

Proof :



≥ 0 by defn
≤ 0 because internal flow cancels out

Vertices from which s is reachable in residual network

Lemma : If $u_f(x,y) > 0$, then $h(x) \leq h(y)+1$

Proof : Initially, edges into s go uphill ; all other edges are flat.

Push $x \longrightarrow y$ $h(x) \geq h(y)$
New (y,x) edge goes uphill.

Relabel
x $h'(x) = h(x)+1$
$\forall (x,y)$, either $u_f(x,y) = 0$
or $h(y) \geq h(x)$
$\Rightarrow h'(y) = h(y) \geq h(x) = h'(x)-1$

Proof of lemma 1 : If the height of a vertex is increased, it has excess flow >0
$\Rightarrow \exists$ $v-s$ path in residual network
$\Rightarrow h(v) \leq h(s) +$ length of path
$\leq |V| + |V|-1 = 2|V|-1$
(applying after the increase in height)

Proof of lemma 2 : There must be ≥ 2 relabels on x between two saturating pushes on (x,y) since there must be a push on (y,x) in between.
$\Rightarrow$ # of saturating pushes on $(x,y) \leq |V|$

Proof of lemma 3 :

Potential $\Phi = \sum_{v : excess(v) > 0} h(v)$

$\Phi_{init} = 0$
$\Phi_{final} = 0$ (only t has excess, but $h(t) = 0$)

Relabel : $\Delta\Phi = 1$
Saturating push on (x,y) : $\Delta\Phi \leq 2|V|$ (by creating excess on y)
Non-saturating push : $\Delta\Phi \leq h(y) - h(x)$

on (x,y) ≤ -1

# of non-saturating pushes

$$\leq (2|V|-1)(|V|-2)$$

$$+ \ 2|V| \ (2|V||E|)$$

$$\leq 4|V|^2|E|$$

We have shown that total # of opns

$$= O(|V|^2|E|)$$

Push out of max height $= O(|V|^2\sqrt{|E|})$

Dynamic tree with simultaneous pushes

$$= O(|V||E| \log |V|)$$

Best strongly polynomial algo!