Lecture 4.1
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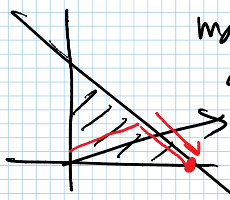
11:13 PM

Linear programming



maximize in direction (2,1), i.e. maximize $(x,y)\cdot(2,1) = 2x+y$

$x+y=1$

$$\begin{aligned} &\text{maximize } \boxed{2x+y} \quad \text{objective} \\ &\text{subject to } \boxed{\begin{aligned} x+y &\le 1 \\ x &\ge 0 \\ y &\ge 0 \end{aligned}} \quad \text{constraints} \end{aligned}$$

In linear programming, both objective and constraints are linear in the variables. The constraints define "half-spaces" and are tight on the corresponding "hyperplanes". The intersection of these half-spaces define a "polyhedron", which is convex (thus, LP is a special case of convex programming).

Example: Max flow:

$$\text{Maximize } \sum_x f(s,x) - \sum_y f(y,s)$$

$$\begin{aligned} \text{subject to } & f(x,y) \le u(x,y) \\ & \sum_y f(x,y) - \sum_w f(w,x) = 0 \quad \forall x \ne s,t \\ & f(x,y) \ge 0 \quad \forall (x,y) \in E \end{aligned}$$

Canonical forms of LP

$$\begin{aligned} \min\ & c\cdot x \\ \text{s.t. } & Ax \ge b \\ & x \ge 0 \end{aligned} \qquad\qquad \begin{aligned} \max\ & c\cdot x \\ \text{s.t. } & Ax \le b \\ & x \ge 0 \end{aligned}$$

Any LP can be converted to canonical form.
How would you do it for the maximum flow LP?

Primal-dual pairs

Motivation: Suppose you have a minimization LP:

$\min c\cdot x$] let x^* be optimal solution

$$\min\ c\cdot x$$
$$Ax \geq b$$
$$x \geq 0$$
} let x^* be optimal solution

How would you show that $c.x^* \geq \alpha$ for some α?
- Demonstrate feasible solution x with $c.x = \alpha$.

How would you show that $c.x^* \leq \alpha$ for some α?
- Harder because you can't demonstrate a solution

Suppose you find a y s.t.
$$A^T y \leq c^T,\ y \geq 0$$

Claim: $b^T y$ is a lower bound on cx^*

Proof: We have
$$Ax^* \geq b$$
$$\Rightarrow\ y^T A x^* \geq y^T b \quad (\text{using } y \geq 0)$$
$$\Rightarrow\ \underbrace{y^T A}_{\leq c} x^* \geq y^T b$$
$$\Rightarrow\ c x^* \geq y^T b = b^T y \quad (\text{using } y^T A \leq c)$$

Clearly, the larger the value of $y^T b$, the better the lower bound is. We call this LP the dual of the previous LP.
$$\max\ b^T y$$
$$\text{subject to}\quad A^T y \leq c^T$$
$$y \geq 0$$

We have already shown weak duality, i.e. dual objective (max) is a lower bound on primal objective (min). Similarly, dual objective (min) is an upper bound on primal objective (max). In fact, duals are reversible, dual of dual is primal.

Mechanical procedure for obtaining duals

A different flow LP based

on flow decomposition.

$$\max \sum_{p \in P(s,t)} f_p \longrightarrow \ell(x,y)$$

s.t. $\sum_{p:(x,y)\in p} f_p \le u(x,y) \quad \forall (x,y)\in E$

$f_p \geq 0$

$$\min \sum_{(x,y)\in E} u(x,y)\, \ell(x,y)$$

s.t. $\sum_{(x,y)\in p} \ell(x,y) \geq 1 \quad \forall p \in P(s,t)$

$\ell(x,y) \geq 0$

Dual: Length function ℓ of min volume s.t. $d_\ell(s,t) \geq 1$.

For any (s,t) cut $(S,\bar{S})$, $\ell(x,y) = \begin{cases} 1 & \text{if } (x,y)\in(S,\bar{S}) \\ 0 & \text{o.w.} \end{cases}$

is feasible and therefore max flow $\leq$ min cut by weak duality!

<u>Implications of weak duality:</u>

For a primal-dual pair P, D either of the following must hold:

(1) P and D are feasible and bounded

(2) One of P and D is feasible but unbounded; then the other must be infeasible.

(3) Both P and D are infeasible.

In the first case, we can say more about the objective values.

<u>Strong Duality Theorem</u>

<u>Thm</u>: If P and D are feasible, x^* is optimal for P iff

(1) x^* is feasible for P, and

(2) $\exists y^*$ feasible for D s.t. $c x^* = y^{*T} b$

[Here, we are using the standard form primal-dual pair:

$$\min\ c x \qquad\qquad \max\ b^T y$$

$$\text{s.t. } Ax = b \qquad\qquad \text{s.t. } A^T y \leq c^T$$

$x \geq 0$

To prove this theorem, we will need the separating hyperplane Thm.

Thm: If x is not inside a closed convex set C, then $\exists$ a hyperplane separating x from C.

We use this thm to first show Farkas Lemma.

Farkas Lemma: Exactly one of the following is feasible:

(1) $\{x : Ax = b : x \geq 0\}$ or

(2) $\{y : A^T y \leq 0,\ y^T b > 0\}$

Proof: If (1) is feasible, then feasibility of (2) will violate weak duality of the pair:

$$\min 0 \quad \text{s.t. } Ax = b,\ x \geq 0 \qquad\qquad \max b^T y \quad \text{s.t. } A^T y \leq 0$$

Conversely, if (1) is not feasible, $b \notin \{Ax : x \geq 0\}$

By separating hyperplane thm,

$$\exists y \text{ s.t. } y^T \cdot b > \sup_{x : x \geq 0} \{y^T (Ax)\}$$

$$\Rightarrow b^T y > \sup_{x : x \geq 0} (x^T A^T y) \geq 0 \text{ since } x \text{ can be } 0$$

If $A^T y > 0$, then $x^T A^T y$ is unbounded for $x \geq 0$. Thus $A^T y \leq 0$.

Finally, we prove strong duality thm using Farkas Lemma.

If the two conditions are met, then by weak duality x^* is optimum.

... we want to show then

Conversely, let x^* be optimum; ...

$$\left\{\underbrace{Ax=b}_{y'},\ \underbrace{cx \le b^T y}_{\tau},\ \underbrace{A^T y \le c}_{x'},\ x\ge 0\right\} \text{ infeasible}$$

weak duality gives the other direction

By Farkas' lemma,

$$\exists\ y',\ x'\ge 0,\ \tau \ge 0 \quad \begin{cases} A^T y' - c\tau \le 0 \\ Ax' - b\tau = 0 \\ b^T y' - c^T x' = 1 \end{cases}$$

- $\tau = 0$ is violated by weak duality
- if $\tau > 0$,

$$0 \ge (-y')^T(Ax' - b\tau) + x'^T(A^T y' - c\tau)$$
$$= \tau(b^T y' - c^T x') = \tau$$

Complementary Slackness

Either constraint is tight or variable is 0 at optimality, i.e.

i.e. $y^{*T}(Ax^* - b) = 0$

Proof: $y^{*T}(Ax^* - b) = (y^{*T}A)x^* - y^{*T}b$

$$= x^{*T}(A^T y^*) - b^T y^*$$
$$\le x^{*T} c^T - b^T y^*$$

$= c^T x^* - b^T y^*$

$= 0$ by strong duality