Applications of LP Duality

I. Flow - cut duality

$$\max \sum_{p \in P(s,t)} f_p \qquad\qquad \min \sum_{e \in E} l_e u_e$$

$$\sum_{p : e \in p} f_p \leq u_e \;\forall e \in E \qquad\qquad \sum_{e : e \in p} l_e \geq 1 \quad \forall p \in P(s,t)$$

$$f_p \geq 0 \qquad\qquad l_e \geq 0$$

Weak Duality $\Rightarrow$ Max flow $\leq$ Min cut

Theorem: Max flow = Mincut

Proof Let l_e^* be the optimal length function.

Define $S_\rho = \{ v : d_{l^*}(u,v) \leq \rho \}$

Let $E(S_\rho) = \sum_{e : e \in (S_\rho, \bar{S}_\rho)} u_e$

Let $\rho_{av} \in_{uar} (0,1)$. Then,

$$\mathbb{E}x\left[E(S_{\rho_{av}}) \right] = \sum_{xy \in E} u_{xy} \Pr\left[xy \in (S_{\rho_{av}}, \bar{S}_{\rho_{av}}) \right]$$

$$\leq \sum_{xy \in E} u_{xy} \Pr\left[d_{l^*}(s,x) \leq \rho_{av} ; d_{l^*}(s,y) > \rho_{av} \right]$$

$$\leq \sum_{xy \in E} u_{xy} \, l^*_{xy}$$

$$\Rightarrow \min_{\rho} \{E(S_\rho)\} \le \mathbb{E}_x [E(S_{\rho_{av}})] \le \sum_{xy \in E} u_{xy} l_{xy}$$

Thus, mincut LP is integral and mincut = maxflow.

Global cuts

Directed

$$\min \sum_{e \in E} x_e$$

$$\sum_{e \in T} x_e \ge 1 \quad \forall \text{ r-arborescences } T \in A_r$$

$$x_e \ge 0$$

$$\max \sum_{T \in A_r} l_T$$

$$\sum_{T : e \in T} l_T \le 1$$

$$l_T \ge 0$$

Theorem (Edmonds): The minimum r-cut is equal to the maximum number of edge disjoint r-arborescences, i.e. the LPs are integral.

Undirected:

$$\min \sum_{e \in E} x_e$$

$$\sum_{e \in T} x_e \ge T \quad \forall \text{ spanning trees } T \in \mathcal{J}$$

$$x_e \ge 0$$

$$\max \sum_{T \in \mathcal{J}} y_T$$

$$\sum_{T : e \in T} y_T \le 1 \quad \forall e \in E$$

$$y_T \ge 0$$

These LPs are not integral!

Theorem (Nash-Williams - Tutte): The maximum number of edge-disjoint spanning trees in an undirected graph is at least half of the minimum global cut.

Bipartite Matchings

Bipartite Matchings

$$\min \sum_{e \in E} x_e$$

$$\sum_{e \in E_v} x_e \le 1$$

$$x_e \geq 0$$

$$\max \sum_v w_v$$

$$w_x + w_y \geq 1 \quad \forall (x,y) \in E$$

$$w_x \geq 0$$

$\Downarrow$

min vertex cover

Weak duality $\Rightarrow$ Max BM $\le$ Min VC

Theorem: For bipartite graphs, both LPs are integral

Proof: BM: If $x_e = 0$ for some edge, remove it and use induction.

If $x_e = 1$ for some edge, remove its ends and use induction.

Suppose $0 \le x_e \le 1$ for all edges.

Case 1: If there is a cycle, then it must be an even cycle.

Shift ε mass until an edge is 0 or 1.

Case 2: If graph is acyclic, take any leaf - leaf path and shift ε mass until an edge is 0 or 1.

VC: If some vertex has $y_v = 0$, remove it, set other endpoint of all edges incident on it to 1, remove all these vertices and incident edges, and use induction

Similar if some vertex has $y_v = 1$.

So, we have $0 \le y_v \le 1$ for all vertices v.

Suppose $|X| \geq |Y|$ (wlog). Change

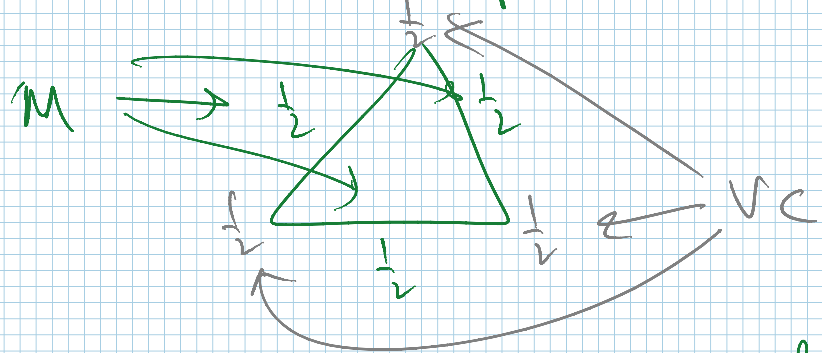
$$y_v \leftarrow y_v + \varepsilon \quad \forall v \in X$$

$$y_v \leftarrow y_v - \varepsilon \quad \forall v \in Y.$$

until some vertex has $y_v = 0$ or $y_v = 1$.

Matching in general graphs

LP is not integral, e.g.



Frac. Mat. = Frac. VC = 3/2

Int. Mat. = 1

Int. VC = 2

Matching polytope: convex hull of all matchings.

Above fractional solution not in matching polytope, i.e. cannot be produced by convex combination of integral matchings.

Edmonds Matching LP:

$$\max \sum_{e \in E} x_e$$

$$\sum_{e \in E_v} x_e \leq 1$$

$$\sum_{e \in S \times S} x_e \leq \left\lfloor \frac{|S|}{2} \right\rfloor$$

← odd set or blossom constraint (non-degenerate only for odd $|S|$)

Theorem (Edmonds): The above LP is integral.