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3:47 PM

Min cost Flow

Find a flow of value f from s to t of minimum cost.
Each edge has cost and capacity.
Min cost circulation: Find a circulation (inflow = outflow) at every vertex of minimum cost.

Lemma: The above problems are equivalent.

Proof: Circulation $\rightarrow$ Flow: $f = 0$

Flow $\rightarrow$ Circulation: Edge (t, s) of capacity mU and cost $-(C+1)n$

We will focus on the min cost circulation problem.

Residual Network: Edge (y, x) of cost $-c(x,y)$ and capacity $f(x,y)$; Edge (x,y) of cost $c(x,y)$ and capacity $u(x,y) - f(x,y)$

Lemma: No negative cycle in residual network $\Leftrightarrow$ Min cost circulation

Proof: $\exists$ neg. cycle $\rightarrow$ can be added to circulation

Circulation not min $\rightarrow$ $f^* - f$ can be decomposed into cycles, one of which must be negative.

(Note: Any circulation can be decomposed into cycles since it can be thought of as s-s flows.)

A dual view

$$\min \sum_{(x,y) \in E} c_{xy} f_{xy}$$

$$\sum_{y} f_{xy} = \sum_{w} f_{wx}$$

$$\max \sum c'_{xy} u_{xy}$$

$$c'_{xy} \leq \underbrace{\pi_x + c_{xy} - \pi_y}_{\text{reduced cost}}$$

prices

$$\sum_y f_{xy} = \sum_w f_{wx}$$

$$f_{xy} \le u_{xy}$$

$$f_{xy} \geq 0$$

↑ prices

$$c'_{xy} \le 0$$

<u>Lemma</u>: No neg. cycle in residual network $\iff \exists \pi$ s.t. $c'_{xy} \geq 0$ $\forall (x,y) \in E$

<u>Proof</u>: no neg cycle in residual network $\iff OPT_{primal} = 0 \iff OPT_{dual} = 0 \iff c'_{xy} = 0\ \forall (x,y) \in E$

$\iff \pi_x + c_{xy} - \pi_y \geq 0\ \forall (x,y) \in E$

<u>Algorithms</u>

1. Cycle canceling: Find negative cycle (using Bellman Ford) and saturate. Continue in residual network.

Running time = $O(mn) \times mUC$

Bellman-Ford

min cost of circulation is $-mUC$

Other combinatorial algorithms use capacity or cost scaling. The first strongly polynomial algorithm was given by Tardos in 1985.

<u>LP-based algorithm for MCF</u>

$$\min \sum c_e f_e$$

$$\sum_{e \in \delta_{in}(v)} f_e - \sum_{e \in \delta_{out}(v)} f_e = b_v$$

$$0 \le f_e \le u_e$$

Lemma: Either $f_e = 0$ or $f_e = u_e$ for some edge or $|E| \leq |V| - 1$ in an extreme point solution.

Proof: Follows from:

(1) $|E|$ linearly independent constraints are tight at an extreme point.

(2) $\sum$ Incidence vector of constraints $= 0 \Rightarrow \leq |V| - 1$ of these constraints are linearly independent.

Iterative Algo:

(1) $f_{xy} = 0$: remove (x, y)

(2) $f_{xy} = u_{xy}$: remove (x, y) and set $b_x \leftarrow b_x + u_{xy}$; $b_y \leftarrow b_y - u_{xy}$

(3) $|E| \leq |V| - 1 \Rightarrow \exists\, x$ s.t. degree$(x) \leq 1$. If degree$(x) = 0$, remove x. If degree$(x) = 1$, remove (x, y) or (y, x) and update b_y accordingly.