Monte Carlo Algorithms

- Algorithm "succeeds" whp (usually $1 - 1/n^d$) for any constant.

Contraction Algorithm for Global Min cut

$G_n = G$

for $i = n-1$ to 2 do
- contract an edge chosen uniformly at random in G_{i+1} to produce G_i
- remove self-loops if any

Output cut represented by two supervertices in G_2

Analysis

Lemma: For any given global mincut $(S, \bar{S})$,

$$P[(S,\bar{S}) \text{ is output}] = \Omega(1/n^2)$$

Proof: $P[(S,\bar{S}) \text{ is a cut in } G_i \mid (S,\bar{S}) \text{ is a cut in } G_{i+1}]$

$$\geq 1 - \frac{\lambda}{(i+1)(\lambda/2)} \quad \text{since deg. of any vertex} \geq \lambda$$

$$= \frac{i-1}{i+1}$$

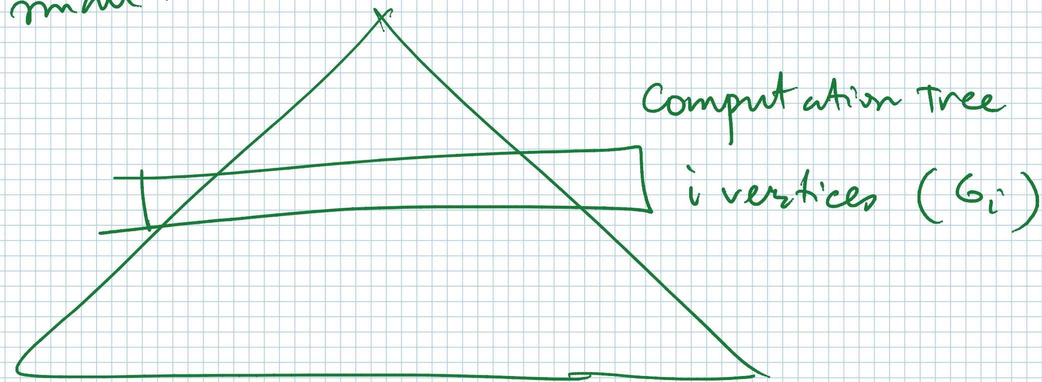
$$P[(S,\bar{S}) \text{ is output}) \geq \prod_{i=n-1}^{2} \frac{i-1}{i+1} = \frac{1 \cdot 2}{n(n-1)} = \Omega(1/n^2)$$

How do we obtain a high probability bound?
- Repeat $O(n^2 \lg n)$ times and output the smallest cut.

Running time: The contraction algorithm can be implemented in $\tilde{O}(n^4)$ time.

Improving to an $\tilde{O}(n^2)$ algorithm:

Intuition: Prob. of success decreases when the number of vertices is small.



How many independent trials for G_i? Call this N_i.

Recall: $\mathbb{P}[(S, \bar{S}) \text{ survives in } G_i] = \Theta\left(\frac{i^2}{n^2}\right)$

$\mathbb{P}[(S, \bar{S}) \text{ survives in } G_i \text{ for some trial}] = 1 - \left(1 - \frac{i^2}{n^2}\right)^{N_i}$

Set $N_i = \left(\frac{n^2}{i^2}\right) \log n$.

What is the size of the computation tree?

$$\sum_{i=n-1}^{2} N_i = O(n^2 \log n)$$

Why is this not formally correct? N_i iterations NOT independent!

Formally, run contraction algo till $i = n/\sqrt{2}$, spawn two copies G_1 and G_2 and return mincut among those reported recursively by G_1 and G_2.

— $T(n) = 2T(n/\sqrt{2}) + O(n^2) \Rightarrow T(n) = O(n^2 \log n)$

— Prob. of success $\geq 1/\log n$.

Proof: Recursion: Think of $d = \log n$ depth binary tree with unbiased coin toss at each node

What is the probability of a path of all Heads?

$$P_d \geq \frac{1}{2} \cdot \left(1 - \left(1 - P_{d-1}\right)^2\right)$$

Now,

$$\frac{1}{2}\left(1 - \left(1 - \frac{1}{d-1}\right)^2\right)$$

$$= \frac{1}{2} \frac{2d-3}{(d-1)^2} \geq \frac{1}{d}$$

Prob. of success $= 1/\log n$