A problem is strongly NP-hard if it does not admit pseudo-polynomial algorithms.

PTAS : $(1 \pm \varepsilon)$-approx in $f(|I|, 1/\varepsilon)$ time where f is poly for fixed ε.

FPTAS : Special case of PTAS where $f = \text{poly}(|I|) \cdot \text{poly}(1/\varepsilon)$

Knapsack

- n items with weight w_i, profit p_i for the ith item
- Knapsack of capacity W
- Find subset of items of maximum value whose total weight is $\leq W$
- One of Karp's 21 NP-hard problems
  (3-SAT $\longrightarrow$ Knapsack)
- Also called 0-1 knapsack; fractional version where fractions of items can be picked has an exact greedy algorithm
- Pseudo-poly algo: Dynamic program
  $A(i, p)$ = min weight combination among first i items that has overall profit of p.

$$A(i+1, p) = \begin{cases} A(i, p) & \text{if } p_{i+1} > p \\ \min\left(A(i,p),\ A(i, p - p_{i+1}) + w_{i+1}\right) & \text{o.w.} \end{cases}$$

# of entries : $O(n^2 P)$ where $P = \max_i \{p_i\}$

How to convert to an FPTAS:

$K = \dfrac{\varepsilon P}{n}$; round all profits to multiples of K

$$\left.\text{Error} \leq \frac{\varepsilon P}{n} \cdot n = \varepsilon P \right\} \ (1-\varepsilon)\text{-approx}$$

$\text{Error} \le \frac{\varepsilon P}{n} \cdot n = \varepsilon P$ $\Big\}$ $(1-\varepsilon)$- approx

$OPT \geq P$

# of entries in DP table: $O\left(n^2 \frac{P}{K}\right) = O\left(n^3/\varepsilon\right)$

Bin packing and Load Balancing

Bin packing: Items of sizes $a_1, a_2, \ldots, a_n$ to be packed in minimum number of bins of unit size.

If $a_i \le 1$ feasible, else infeasible

First-fit: Pack items in arbitrary order into existing bins; if not possible, start a new bin.

Analysis: At most one bin is less than $\frac{1}{2}$ full.

Better Algo:

If there are K item types and each item is of size $\geq \varepsilon$, then the number of configurations of a bin is $K^{1/\varepsilon}$. Since there are at most n bins, number of configurations overall is $n^{K^{1/\varepsilon}}$.

To ensure this, remove all items of size $< \varepsilon$ and round all other items into equi-sized groups containing $n\varepsilon^2$ items each. Then, round the sizes of all items to the max size its group.

Lemma: OPT changes by a factor of $1+\varepsilon$.

Proof: In original OPT, replace each item of class i by a rounded item of class $i-1$. Place $n\varepsilon^2$ items of highest class to private bins. Finally note that $OPT \geq n\varepsilon$.

To complete the algo, use FIRSTFIT for the items of size $\le \varepsilon$.

If new bins are used, all except one bin must be full upto a volume of $(1-\varepsilon)$.

Running time : $n^{(1/\varepsilon^2)^{1/\varepsilon}}$