We conclude our study of LP rounding with a simple but surprisingly powerful algorithm for the MAX SAT problem.

Problem: Given a CNF formula, find an assignment that maximizes the number of satisfied clauses.

Algorithm: Choose each variable independently at random to be 1 w/p $\frac{1}{2}$. Then, if clause i has k literals,

$$P_i = \Pr[\text{clause } i \text{ is satisfied}] = 1 - 2^{-k} \geq \frac{1}{2}$$

Dealing with small clauses

$$\max \sum_i z_i$$

$$\sum_{j \in V_i^+} y_j + \sum_{j \in V_i^-} (1 - y_j) \geq z_i \qquad \forall\, i \in C$$

$$0 \leq z_i \leq 1 \qquad \forall\, i \in C$$

Rounding:
$$x_j = \begin{cases} 1 & \text{w/p} \quad y_j^* \\ 0 & \text{o.w.} \end{cases}$$

Lemma: If clause i has k literals, then
$$P_i = \Pr[\text{clause } i \text{ is satisfied}] \geq \left(1 - \frac{1}{e}\right) z_i^*$$

Proof: $P_i = 1 - \prod_{j \in V_i^+} (1 - y_j^*) \prod_{j \in V_i^-} y_j^*$

Let
$$y_j^{(i)*} = \begin{cases} 1 - y_j^* & \text{if } j \in V_i^- \\ y_j^* & \text{if } j \in V_i^+ \end{cases}$$

Then $P_i = 1 - \prod (1 - y_j^{(i)*}) \leq 1 - \left(1 - \sum y_j^{(i)*}\right)^k$

Then $p_i = 1 - \prod_{j \in V_i} (1 - y_j^{(i)*}) \geq 1 - \left(1 - \frac{\sum y_j^{(i)*}}{k}\right)^k$

$$\geq 1 - \left(1 - \frac{z_i^*}{k}\right)^k \geq \left(1 - \left(1 - \frac{1}{k}\right)^k\right) z_i^*$$

$$\geq \left(1 - \frac{1}{e}\right) z_i^*$$

So, this algorithm has an approximation factor of $1 - \frac{1}{e}$.

We now combine the above two algorithm to obtain a better approx ratio. This combination is inspired by the observation that for long clauses, the naive greedy algorithm outperforms the LP rounding algorithm and vice-versa for short clauses.

Algo: Run both algorithms and choose the better solution.

Analysis: This algo is at least as good as running naive greedy with probability p and LP rounding w/p $(1-p)$ for any probability p.

Then, $p_i \geq \left(p \cdot (1 - 2^{-k}) + (1-p)\left(1 - \left(1 - \frac{1}{k}\right)^k\right)\right) z_i^*$

Choose $p = \frac{1}{2}$. Then

$$p_i \geq \frac{1}{2}\left(2 - 2^{-k} - \left(1 - \frac{1}{k}\right)^k\right) z_i^*$$

Let $\alpha(k) = 2^{-k} + \left(1 - \frac{1}{k}\right)^k \leq \frac{1}{2}$

$\Rightarrow p_i \geq \frac{1}{2}\left(2 - \frac{1}{2}\right) z_i^* = 3/4 \; z_i^*$