COMPSCI 330 Lecture 19 Linear Programming

Monday, November 7, 2016 2:23 PM

- Linear Programming
- Canonical Form
- Formulating problems as Linear Programs

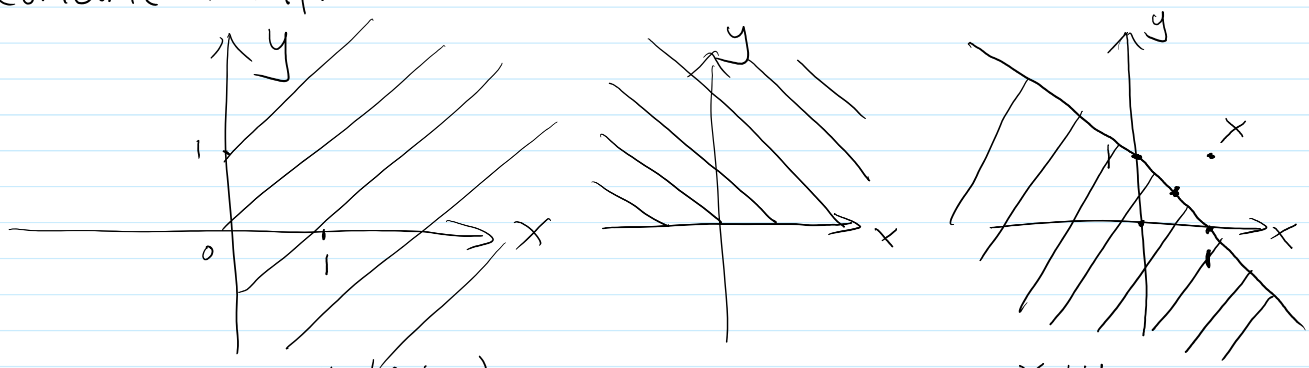
- recall: graph algorithm shortest path, MST, bipartite matching
- LP (linear programming) general tool for solving many problems
- system of linear inequalities
- example

$$\max\ 2x+y \leftarrow \text{objective}$$

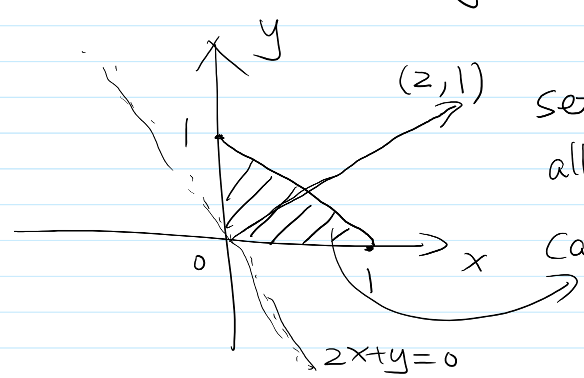
constraints $\left\{ \begin{array}{l} \text{s.t. } x \geq 0 \\ y \geq 0 \\ x+y \leq 1 \end{array} \right.$

linear function: polynomial of degree 1 (does not have x^2, y^2, xy terms)

- geometric interpretation



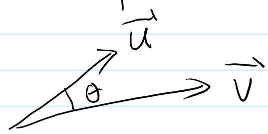
$x \geq 0$ (halfplane) $\quad y \geq 0 \quad x+y \leq 1$



set of points that satisfy all of the linear constraints called "feasible solutions"

objective: $\max\ 2x+y$

(recall: inner product $(x,y)\cdot(x',y') = x\cdot x' + y\cdot y'$



$\vec{u}\cdot\vec{v} = \|u\|\cdot\|v\|\cdot\cos\theta$

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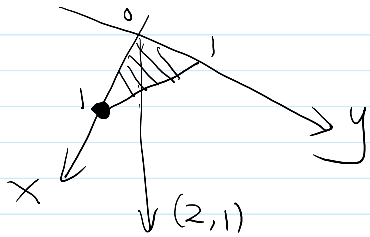


- linear programming is equivalent to finding "lowest point" in the feasible set.

- optimal solution: $x=1$ $y=0$ $2x+y=2$
- in higher dimensions (more than 2 variables), feasible solutions form a polytope (polyhedral), harder to visualize

- canonical form of linear Program
  - Def: a LP in canonical form looks like

$$\begin{aligned} \min\ & c^T x \\ \text{s.t.}\ & Ax \geq b \\ & x \geq 0 \end{aligned} \qquad \begin{aligned} x &\in \mathbb{R}^n \quad c \in \mathbb{R}^n \\ A &\in \mathbb{R}^{m \times n} \\ b &\in \mathbb{R}^m \end{aligned}$$

$$\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{x} \text{ n variables (unknown)} \qquad \underbrace{\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}}_{c}$$

$c^T x = \sum_{i=1}^{n} c_i x_i$ linear function over x_i

$$m \underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & & & \\ \vdots & & A & \\ a_{m1} & \cdots & \cdots & a_{mn} \end{bmatrix}}_{n} \underbrace{\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}}_{x} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$$

$y_i = \sum_{j=1}^{n} a_{ij} x_j$

$x \geq 0$ means $\forall i,\ x_i \geq 0$

$Ax \geq b$ means $\forall j$ $(Ax)_j \geq b_j$

- expressing the example in canonical form

$$\begin{aligned} \max\ & 2x + y \\ \text{s.t.}\ & x \geq 0 \\ & y \geq 0 \\ & x + y \leq 1 \end{aligned}$$

$x_1 = x$ $x_2 = y$ $\rightarrow$ $\min\ -2x - y$

$c = (-2,-1)^T$ equivalent.

$c^T x = -2x_1 - x_2$

$y \geq 0$ $\rightarrow$ already included in canonical form

$x + y \leq 1$ $\rightarrow$ $-x - y \geq -1$ $\quad A = (-1,-1)$ $\quad b = -1$

$$\min\ c^T x = (-2,-1)x = -2x_1 - x_2$$

$$\text{s.t.} \quad x \geq 0 \iff x_1 \geq 0,\ x_2 \geq 0$$

$$(-1,-1)\,x \geq -1 \iff -x_1 - x_2 \geq -1$$

- Example:

$$\min\ 3x_1 - x_2 + x_3$$

$$\underline{x_1 + 2x_2 = 5} \quad \text{equality constraints}$$
$$x_1 - x_3 \leq 4$$
$$x_2 + x_3 \leq 6$$
$$x_1 \geq 0$$
$$x_2 \geq 0$$
(don't have $x_3 \geq 0$)

equality constraint $\Longrightarrow$ 2 inequality constraints

$$x_1 + 2x_2 \geq 5$$
$$x_1 + 2x_2 \leq 5$$

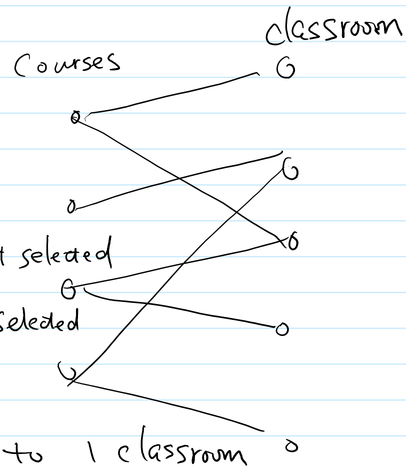
unconstrained variable (x_3) $\Longrightarrow$ use 2 extra variables

$$x_3 = x_4 - x_5 \qquad \begin{pmatrix} x_3 = 2 & x_4 = 2 & x_5 = 0 \\ x_3 = -3 & x_4 = 0 & x_5 = 3 \end{pmatrix}$$
$$x_4 \geq 0$$
$$x_5 \geq 0$$

$$\min\ 3x_1 - x_2 + x_4 - x_5$$
$$x_1 + 2x_2 \geq 5$$
$$-x_1 - 2x_2 \geq -5$$
$$-x_1 + x_4 - x_5 \geq -4$$
$$-x_2 - x_4 + x_5 \geq -6$$
$$x_1, x_2, x_4, x_5 \geq 0$$

$$\underbrace{\begin{pmatrix} 1 & 2 & 0 & 0 \\ -1 & -2 & 0 & 0 \\ -1 & 0 & 1 & -1 \\ 0 & -1 & -1 & 1 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \\ x_4 \\ x_5 \end{pmatrix} \geq \underbrace{\begin{pmatrix} 5 \\ -5 \\ -4 \\ -6 \end{pmatrix}}_{b}$$

- Solving problems using LP.
  - bipartite matching



- select variables

  for edge u,v, $x_{u,v} = 0$ ((u,v) not selected)

  $x_{u,v} = 1$ ((u,v) is selected)

- constraints

  ① each course can only be assigned to 1 classroom

$$\forall u \quad \sum_v x_{u,v} \leq 1$$

② each classroom can only be assigned to 1 course

$$\forall v \quad \sum_u x_{u,v} \le 1$$

③ each edge = 1 if selected, 0 if not selected. (not a linear constraint)

$$0 \le x_{u,v} \le 1 \qquad \text{(relaxed constraint)}$$

- objective function: maximize number of assigned courses/classroom

$$\max \sum_{(u,v)\in E} x_{u,v}$$

- $$\boxed{\begin{aligned} &\max \sum_{(u,v)\in E} x_{u,v} \\ &\forall u \quad \sum_v x_{u,v} \le 1 \\ &\forall v \quad \sum_u x_{u,v} \le 1 \\ &\forall (u,v)\in E \quad 0 \le x_{u,v} \le 1 \end{aligned}}$$

Linear program for bipartite matching.

- Problem: solution to LP may have $x_{u,v} \neq 0, 1$ ($x_{u,v} = \frac{1}{3}, \frac{1}{2}, \ldots$)
- Luckily for this LP, can always find a solution that only has $x_{u,v} = 0/1$ (not true in general)