COMPSCI 330 Lecture 20 LP Duality

Wednesday, November 9, 2016 2:19 PM

- Linear Programming for Two-Player Games
- Duality for Two-Player Games
- LP Duality

- Two player games

e.g. rock paper scissor

- can represent outcome of RPS as a matrix

$$\begin{array}{c|ccc} & R & P & S \\ \hline R & 0 & -1 & 1 \\ P & 1 & 0 & -1 \\ S & -1 & 1 & 0 \end{array}$$

Bob, Alice, A

entry in matrix: result of this game for Alice

Goal of players: Alice wants to maximize the winning probability

Bob wants to minimize Alice's winning probability.

zero-sum games: 1 player wins = the other player loses

- strategy: probability distribution over actions.

Alice: play R w.p. P_R, P w.p. P_P, S w.p. P_S

Bob: play R w.p. q_R, P w.p. q_P S w.p. q_S

- "value" of the game: $\mathbb{E}[\text{payoff}]$

$$= \sum_{\substack{i \in RPS \\ j \in RPS}} P_i \cdot P_j A_{ij}$$

example: if $P_R = 1$, $q_S = 1$, payoff $= 1$ (Alice always wins)

if $P_R = 1$ $q_R = q_P = \frac{1}{4}$ $q_S = \frac{1}{2}$ payoff $= \frac{1}{4}$

(Alice is expected to get 1 net win in every 4 games)

A B C UNC

A	3	1	−1
B	−2	3	2
C	1	−2	4

Duke

Finding a strategy for Duke: play A w.p. x_1, play B w.p. x_2
play C w.p. x_3

want: no matter what UNC does, always get an expected payoff of at least x_4.

- using linear program:
  - Constraints
    (1) probability constraints

$$x_1, x_2, x_3 \geq 0$$
$$x_1 + x_2 + x_3 = 1$$

(2) payoff constraints

$$3x_1 - 2x_2 + x_3 \geq x_4 \qquad \text{UNC plays A}$$
$$x_1 + 3x_2 - 2x_3 \geq x_4 \qquad \text{B}$$
$$-x_1 + 2x_2 + 4x_3 \geq x_4 \qquad \text{C}$$

(1)

  - objective function

$$\max x_4$$

  - optimal solution $\Leftrightarrow$ best strategy
- UNC's strategy
  - play A w.p. y_1, B w.p. y_2, C w.p. y_3
  - expected payoff (for Duke) y_4 (want to minimize y_4)

$$y_1, y_2, y_3 \geq 0$$
$$y_1 + y_2 + y_3 = 1$$
$$3y_1 + y_2 - y_3 \leq y_4 \qquad \text{(Duke plays A)}$$
$$-2y_1 + 3y_2 + 2y_3 \leq y_4 \qquad \text{B}$$
$$y_1 - 2y_2 + 4y_3 \leq y_4 \qquad \text{C}$$

(2)

$$\min y_4 \qquad \text{(minimize payoff for Duke)}$$

- Q: x_4 be optimal value for (1), y_4 be optimal value for (2) which is larger?

- A: $x_4 = y_4$
- Why? if $x_4 > y_4$ we know if both of them play optimal strategy
  expected payoff $\geq x_4$ (guarantee of first LP)
  expected payoff $\leq y_4$ (guarantee of second LP)

  ("weak" duality)

  if $x_4 < y_4$ contradiction is more complicated. But it's also not possible.

  $x_4 = y_4$ "strong" duality for two-player games.

- Solution to LPs: $x_1 = \frac{9}{19}$ $x_2 = \frac{6}{19}$ $x_3 = \frac{4}{19}$ $x_4 = 1$ indeed the same.
  $y_1 = y_2 = y_3 = \frac{1}{3}$ $y_4 = 1$

- Theorem (minimax theorem) For any two-player zero-sum game, there is always a pair of optimal strategies and a value v. If player A plays optimal strategy, can always guarantee payoff $\geq v$.
  If B plays optimal strategy, can always guarantee payoff $\leq v$.

  (RPS, $v = 0$ Duke-UNC value 1)

- Duality for Linear Program

  recall: canonical form

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax \geq b \\ & x \geq 0 \end{aligned}$$

e.g.

$$\begin{aligned} \min \quad & 2x_1 - 3x_2 + x_3 \\ & x_1 - x_2 \geq 1 && (1) \\ & x_2 - 2x_3 \geq 2 && (2) \\ & -x_1 - x_2 - x_3 \geq -7 && (3) \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Q: how can I convince you that optimal solution has value at least -1.

verify optimal ≤ -1: can be done by a feasible solution
$(x_1 = 4, x_2 = 3, x_3 = 0)$
because this is feasible, optimal solution ≤ -1

idea for prove optimal solution ≥ -1

└ because this is feasible, optimal solution ≤ -1

idea for proving optimal solution ≥ -1

$$2.5 \times (1) + 0.5 \times (3)$$

$$\Longrightarrow 2x_1 - 3x_2 - 0.5x_3 \ge -1$$

$$2x_1 - 3x_2 + x_3 \underset{\substack{\uparrow \\ x_3 \ge 0}}{\ge} 2x_1 - 3x_2 - 0.5x_3 \ge -1$$

optimal solution is at least -1 !

optimal value $= -1$
optimal solution
$x_1 = 4$ $x_2 = 3$ $x_3 = 0$

- general way for finding these proof?
  - idea: can actually find the proof using a LP!

Primal LP

$$\min c^T x$$

s.t.

$a_1^T x \ge b_1$	(1)	y_1
$a_2^T x \ge b_2$	(2)	y_2
$\vdots$	$\vdots$	$\vdots$
$a_m^T x \ge b_m$	(m)	y_m

$x_i \ge 0$

for each constraint, have a variable y_i

$$y_1 x(1) + y_2 x(2) + \cdots + y_m x(m)$$

$$\left(\sum_{i=1}^m y_i a_i\right)^T x \ge \left(\sum_{i=1}^m y_i b_i\right)$$

want to show $c^T x \ge \left(\sum_{i=1}^m y_i a_i\right)^T x \ge \left(\sum_{i=1}^m y_i b_i\right)$

to show this need $c_j \ge \sum_{i=1}^m y_i a_i(j)$

in order to find best proof of this ~~kid~~ kind

$$\left.\begin{aligned} & y_i \ge 0 \\ & c_j \ge \sum_{i=1}^m y_i a_i(j) \end{aligned}\right\} \text{dual LP}$$

$$\max \sum_{i=1}^m y_i b_i$$

- more succinctly

$$\begin{array}{c} \min c^T x \\ Ax \ge b \\ x \ge 0 \\ \text{primal} \end{array} \iff \begin{array}{c} \max b^T y \\ A^T y \le c \\ y \ge 0 \\ \text{dual} \end{array}$$