COMPSCI 330 Lecture 21 Linear Programming

Monday, November 14, 2016 2:57 PM

- More properties of Linear Programs
- Simplex Algorithm

– Recall

Primal

$$\min\ 2x_1 - 3x_2 + x_3$$

$$x_1 - x_2 \geq 1 \quad (1)$$

$$x_2 - 2x_3 \geq 2 \quad (2)$$

$$-x_1 - x_2 - x_3 \geq -7 \quad (3)$$

$$x_1, x_2, x_3 \geq 0$$

Dual

$$\max\ y_1 + 2y_2 - 7y_3$$

$$y_1 - y_3 \leq 2 \quad (d_1)$$

$$-y_1 + y_2 - y_3 \leq -3 \quad (d_2)$$

$$-2y_2 - y_3 \leq 1 \quad (d_3)$$

$$y_1, y_2, y_3 \geq 0$$

$$\begin{array}{c} \min\ c^T x \\ Ax \geq b \\ x \geq 0 \end{array} \quad \Longleftrightarrow \quad \begin{array}{c} \max\ b^T y \\ A^T y \leq c \\ y \geq 0 \end{array}$$

weak duality: For any feasible solution x for primal, y for dual, we have

$$\underbrace{c^T x}_{\text{value of primal solution}} \underset{①}{\geq} \text{primal OPT} \underset{②}{\geq} \underbrace{b^T y}_{\text{value of the dual solution}}$$

interpretation: primal feasible solutions are upperbounds
dual feasible solutions are lowerbounds
primal feasible $\geq$ dual feasible

Proof: ① trivial. x is a feasible solution, so optimal solution cannot be worse than $c^T x$.

② consider $y_1 \times (1) + y_2 \times (2) + y_3 \times (3)$

$$\underbrace{(y_1 - y_3)}_{(d_1)} x_1 + \underbrace{(-y_1 + y_2 - y_3)}_{(d_2)} x_2 + \underbrace{(-2y_2 - y_3)}_{(d_3)} x_3 \geq y_1 + 2y_2 - 7y_3$$

because y is a dual feasible solution, we know

$$y_1 - y_3 \leq 2 \quad (-y_1 + y_2 - y_3) \leq -3 \quad (-2y_2 - y_3) \leq 1$$

therefore whenever $x_1, x_2, x_3 \geq 0$

$$2x_1 - 3x_2 + x_3 \geq (y_1 - y_3) x_1 + (-y_1 + y_2 - y_3) x_2 + (-2y_2 - y_3) x_3$$

for any feasible solution (satisfy (1) (2) (3))

$$(y_1 - y_3) x_1 + (-y_1 + y_2 - y_3) x_2 + (-2y_2 - y_3) x_3 \geq y_1 + 2y_2 - 7y_3$$

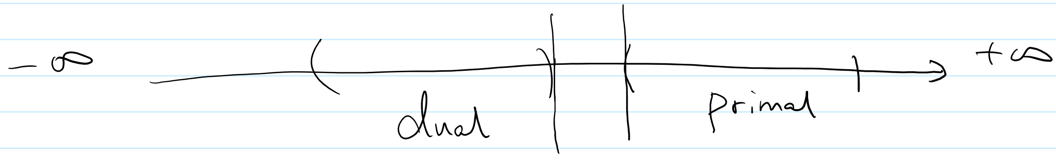
for any feasible solution

for any feasible solution

$$2x_1 - 3x_2 + x_3 \geq y_1 + 2y_2 - 7y_3 \qquad \square$$

Claim: If x is a primal solution, y is a dual solution, $c^Tx = b^Ty$, then both x, y are optimal.

This offers a fast way to certify the result of a LP.

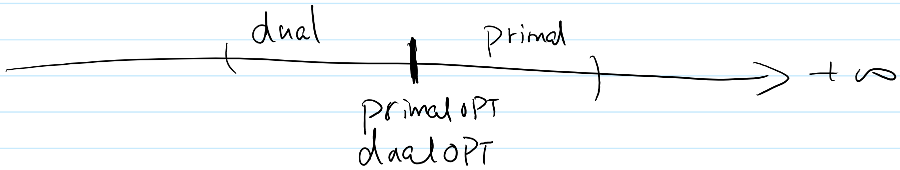


if only weak duality is true

under only weak duality, possible to have primal OPT > dual OPT.

Theorem (Strong duality) For any linear program whose optimal is finite,

$$\text{primal OPT} = \text{dual OPT}.$$



if strong duality is true.

For linear programs, can get a pair of primal and dual optimal solutions.

- Complimentary slackness
  - consider tight constraints for LP

Primal	Dual
$\min\ 2x_1 - 3x_2 + x_3$	$\max\ y_1 + 2y_2 - 7y_3$
$x_1 - x_2 \geq 1$ (1)	$y_1 - y_3 \leq 2$
$x_2 - 2x_3 \geq 2$ (2)	$-y_1 + y_2 - y_3 \leq -3$
$-x_1 - x_2 - x_3 \geq -7$ (3)	$-2y_2 - y_3 \leq 1$
$x_1, x_2, x_3 \geq 0$	$y_1, y_2, y_3 \geq 0$

Primal OPT $x_1 = 4,\ x_2 = 3\ x_3 = 0$ dual OPT $y_1 = 2.5\ y_2 = 0\ y_3 = 0.5$

tight constraints

$$x_1 - x_2 = 1 \quad (1)$$
$$-x_1 - x_2 - x_3 = -7 \quad (3)$$
$$x_3 = 0$$

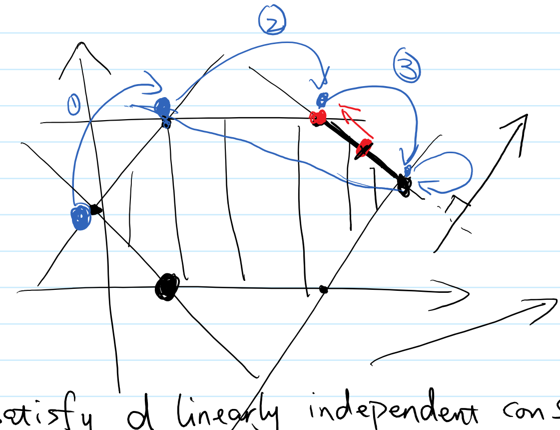
y_1, y_3 are nonzero dual variables.

Lemma: (Complimentary Slackness) Suppose x is primal OPT, there is a dual OPT solution y such that for any constraint i

$$y_i\,(a_i^Tx - b_i) = 0$$

(either $y_i = 0$ or constraint $a_i^T x - b_i \geq 0$ is tight)

- Solving Linear Program
  - recall: interpret the constraints geometrically



  - basic feasible solution
    - geometry: vertex
    - algebra: feasible solution that satisfy d linearly independent constraints

Claim: for any LP, there is always a basic feasible solution that is optimal.

Consider an LP with m constraints and n variables

Claim $\Rightarrow$ there is a $\binom{m+n}{n} n^3$ algorithm for LP

$\binom{m+n}{n}$: # of ways to choose n constraints out of the $m+n$ constraints

this enumerates all subset of n constraints, check if they correspond to a basic feasible solution, then find the best solution among all basic feasible solutions.

- Simplex algorithm:

  idea: start with a basic feasible solution

  in each iteration, try to find an adjacent basic feasible solution with better value.

  - Correctness / running time not required for this course
  - running time: exponential in dimension in worst case

    in practice very fast

  - other algorithms can achieve poly$(m, n, \log\frac{1}{\varepsilon})$

    # constraints # var accuracy