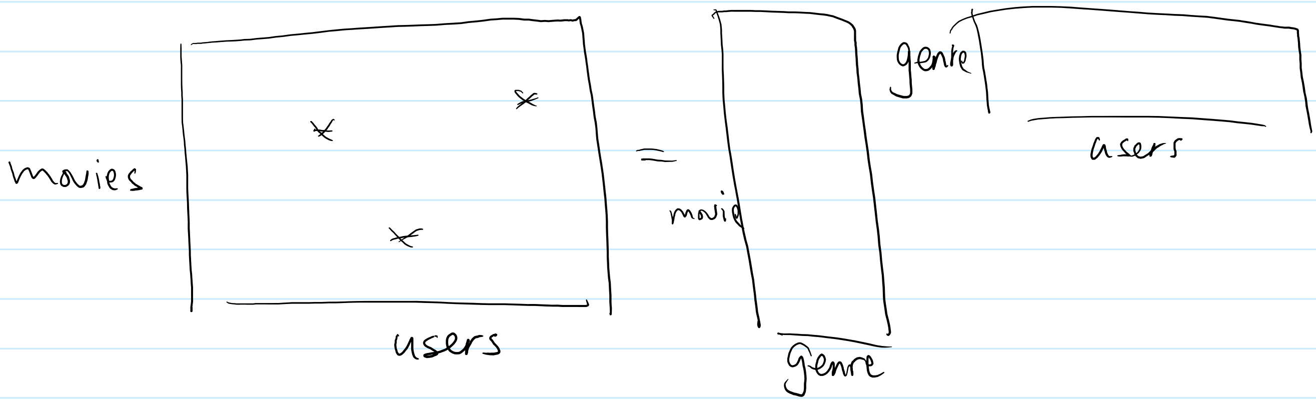
Lecture 18 Matrix Completion

Tuesday, November 1, 2016 8:51 PM

- Matrix Completion
  - recall: netflix challenge



- low rank matrix M, observe subset of entries
- Goal: recover the low rank matrix
- This lecture: $M = zz^T$ for vector z (rank 1)

  will show $f(x) = \|M - xx^T\|_\Omega^2$

  $$= \sum_{(i,j)\in\Omega} \left(M_{ij} - (xx^T)_{i,j}\right)^2$$

  has no bad local optima.

- recall: saddle points, 1st and 2nd order optimality

  $$g(x) = \|M - xx\|_F^2$$

  $$\nabla g(x) = 2(M - xx^T)x$$

  $$\nabla^2 g(x) = 4xx^T - 2M + 2\|x\|^2 \cdot I$$

how to compute? expand

$$g(x+\Delta) = \| M - (x+\Delta)(x+\Delta)^T \|_F^2$$

$$= \|M - xx^T\|_F^2 - \langle M - xx^T, x\Delta^T + \Delta x^T \rangle$$

$$+ \|x\Delta^T + \Delta x^T\|_F^2 - 2\langle M - xx^T, \Delta\Delta^T \rangle$$

- Critical points: $(M - xx^T)x = 0$

$$\Rightarrow x = \pm z \text{ or } x = 0$$

$$\nabla^2 g(x) \succeq 0 \Rightarrow x \neq 0$$

- generalize this to matrix completion?

$$\nabla f(x) = 2(M - xx^T)_\Omega x$$

$$\Delta^T (\nabla^2 f(x)) \Delta = \|x\Delta^T + \Delta x^T\|_\Omega^2 - 2\Delta^T P_\Omega (M - xx^T) \Delta$$

hard to solve for $\nabla f(x) = 0$

- "Simple" proof.

Claim: $\langle \nabla g(x), x \rangle = 0 \Rightarrow \langle x, z \rangle^2 = \|x\|^4$

Proof: $\langle \nabla g(x), x \rangle = 2\langle (zz^T - xx^T)x, x \rangle$

$$= 2(\langle x, z \rangle^2 - \|x\|^4)$$

Claim: $\langle \nabla f(x), x \rangle = 0 \Rightarrow \langle x, z \rangle^2 \geq \|x\|^4 - \varepsilon$

Proof: $\langle \nabla f(x), x \rangle = 2 \underbrace{\langle P_\Omega (zz^T - xx^T), xx^T \rangle}_{\substack{\text{expectation} \\ p\langle zz^T - xx^T, xx^T \rangle}}$

(concentration) $= 2(\langle zz^T - xx^T, xx^T \rangle \pm \varepsilon)$

2nd order

Claim: $z^T \nabla^2 g(x) z \geq 0 \Rightarrow \|x\|^2 \geq \frac{1}{3}$

Proof: $z^T \nabla^2 g(x) z = 4\langle x, z\rangle^2 - 2 z^T (zz^T) z \quad + 2\|x\|^2\|z\|^2$

$$\leq 6\|x\|^2\|z\|^2 - 2\|z\|^4$$

$$\Rightarrow 2\|z\|^4 \leq 6\|x\|^2\|z\|^2$$

$$\|x\|^2 \geq \frac{1}{3}\|z\|^2 = \frac{1}{3}$$

Claim: $z^T \nabla^2 f(x) z \geq 0 \Rightarrow \|x\|^2 \geq \frac{1}{3} - \varepsilon$

Proof: $z^T \nabla^2 f(x) z = \underbrace{\|xz^T + zx^T\|_\Omega^2}_{\approx p\|\ \ \|_F^2} - \underbrace{2z^T P_\Omega(zz^T - xx^T) z}_{\approx p z^T(zz^T - xx^T)z}$

Now: if $\langle x, z\rangle^2 \geq \|x\|^4 - \varepsilon$ and $\|x\|^2 \geq \frac{1}{3} - \varepsilon$

$\langle x, z\rangle^2 \geq \frac{1}{9} - O(\varepsilon)$ (already almost correct)

$0 = \langle \nabla f(x), z\rangle \approx \langle x, z\rangle (\|z\|^2 - \|x\|^2)$

so $\|x\|^2 \approx \|z\|^2$ and $|\langle x, z\rangle| \approx 1$.

- Incoherence and concentration.