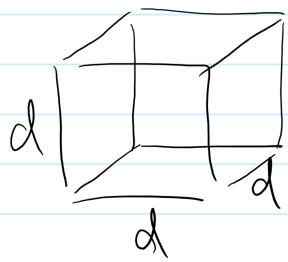
Lecture 8 Tensor Basics

Monday, September 19, 2016 8:36 PM

- Tensors : high dimensional arrays

  $T \in \mathbb{R}^{d\times d\times d}$

  d^3 entries



- Similarity to matrices

matrix M	tensor T
bilinear form $u^T M v = \sum_{i=1}^{d}\sum_{j=1}^{d} M_{ij} u_i v_j$	trilinear form $T(u,v,w) = \sum_{i,j,k} T_{i,j,k} u_i v_j w_k$
symmetric matrix $M_{ij} = M_{ji}$ $\Updownarrow$ degree 2 polynomial $u^T M u = \sum M_{i,i} u_i^2 + 2\sum_{i<j} M_{i,j} u_i u_j$	(super) symmetric tensor $T_{ijk} = T_{ikj} = \cdots = T_{kji}$ $\Updownarrow$ degree 3 polynomial $T(u,u,u) = \sum_{i,j,k} T_{ijk} u_i u_j u_k$
Frobenius norm $\|M\|_F = \sqrt{\sum_{i,j} M_{ij}^2}$	Frobenius norm $\|T\|_F = \sqrt{\sum_{i,j,k} T_{i,j,k}^2}$
Spectral norm $\|M\| = \max_{\|u\|=\|v\|=1} u^T M v$	Spectral norm $\|T\| = \max_{\|u\|=\|v\|=\|w\|=1} T(u,v,w)$
rank-1 matrix uv^T	rank 1 tensor $(u\otimes v\otimes w)_{i,j,k} = u_i v_j w_k$
rank $M = \sum_{i=1}^{r} u_i v_i^T$	rank $T = \sum_{i=1}^{r} u_i \otimes v_i \otimes w_i$

- Difference?

  "rank" of tensor very different from rank of matrix.

  - matrix rank := row rank, column rank
    easy to compute
    stable (rank in $\mathbb{Q}$ same as rank in $\mathbb{R}$
    rank = "border rank")

  - tensor: consider $\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right)$

$$\Updownarrow$$

$$3xy^2$$

    rank = 3

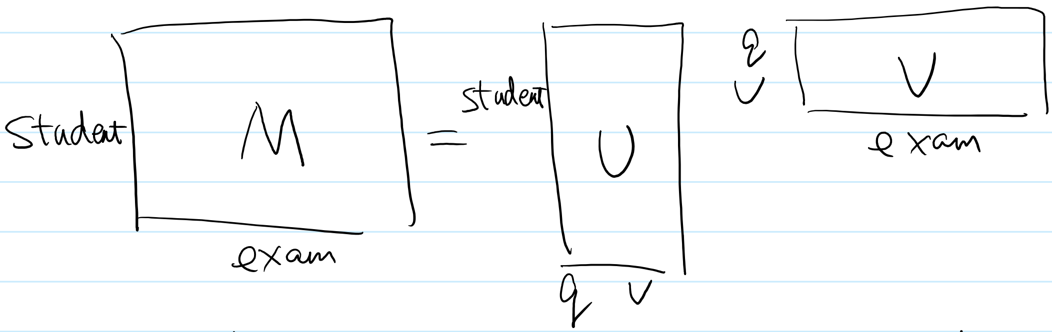
    however: $3xy^2 = \lim_{\varepsilon \to 0} \frac{\overbrace{(x+\varepsilon y)^3 - x^3}}{\underbrace{\varepsilon}_{\text{rank 2 tensor!}}}$

    This is not possible for matrices.

  - NP-hard to compute rank of tensor (or a low rank decomposition)

- Why tensor?

  - Factor Analysis
    - Spearmon's experiments
      - n students, take m different exams
      - hypothesis: two kinds of intelligence (quantitative and verbal)



  - Q: which student has best quatitative intelligence?

- not clear, because $M = UV = (UR)(R^{-1}V)$
- Matrix Decomposition is ambiguous!

- Tensors: consider each exam has two parts: written and oral

$$M_{\text{written}} = U \; V = \lambda_1 u_1 v_1 + \lambda_2 u_2 v_2$$

$$M_{\text{oral}} = U' \; V' = \lambda_1' u_1' v_1' + \lambda_2' u_2' v_2'$$

hypothesis: written/oral does not change students' ability ($u_1 = u_1'$, $u_2 = u_2'$) and exams' relative requirement ($v_1 = v_1'$, $v_2 = v_2'$)

However: verbal skill slightly more important in oral exam $\left(\frac{\lambda_2'}{\lambda_1'} > \frac{\lambda_2}{\lambda_1} \right)$

$$\left(M_{\text{written}}, M_{\text{oral}} \right) = \begin{pmatrix} \lambda_1 \\ \lambda_1' \end{pmatrix} \otimes u_1 \otimes v_1 + \begin{pmatrix} \lambda_2 \\ \lambda_2' \end{pmatrix} \otimes u_2 \otimes v_2$$

- Tensor decomposition $\Longleftrightarrow$ simultaneous matrix decomp

often unique!