COMPSCI330 Design and Analysis of Algorithms
Assignment 0: Solutions

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1 Induction

1. **Induction Hypothesis** Let $P(n) : \sum_{i=0}^{k} 2^i = 2^{k+1} - 1$ be true for all natural numbers $k \leq n \in N$
   We wish to prove $P(n+1)$ holds true,
   Base Case $P(1) : \sum_{i=0}^{1} 2^i = 1 + 2 = 3 = 4 - 1 = 2^{1+1} - 1$ holds true.
   Induction Step

$$\begin{aligned}
\sum_{i=0}^{k+1} 2^i &= 2^{k+1} + \sum_{i=0}^{k} 2^i \\
&= 2^{k+1} + 2^{k+1} - 1 && \text{(from the Induction Hypothesis)} \\
&= 2^{k+2} - 1
\end{aligned}$$

   Thus, we show that P(n+1) holds whenever P(n) holds. Thus, by the principle of Mathematical Induction, $P(n)$ holds for all natural numbers n.

2. **Induction Hypothesis** Let $P(n) : \sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$ be true for all natural numbers $k \leq n \in N$
   We wish to prove $P(n+1)$ holds true,
   Base Case $P(1) : \sum_{i=1}^{1} i^2 = 1 = \frac{1(1+1)(2+1)}{6}$ holds true.

Induction Step

$$\begin{aligned}
\sum_{i=0}^{k+1} i^2 &= (k+1)^2 + \sum_{i=0}^{k} i^2 \\
&= (k+1)^2 + \frac{k(k+1)(2k+1)}{6} && \text{(from the Induction Hypothesis)} \\
&= \frac{(k+1)(6k+6+k(2k+1))}{6} \\
&= \frac{(k+1)(2k^2+7k+6)}{6} \\
&= \frac{(k+1)(2k^2+4k+3k+6)}{6} \\
&= \frac{(k+1)(2k(k+2)+3(k+2))}{6} && \text{(factorising)} \\
&= \frac{(k+1)(k+2)(2k+3)}{6} \\
&= \frac{(k+1)(k+2)(2(k+1)+1)}{6} && \text{(answer)}
\end{aligned}$$

Thus, we show that P(n+1) holds whenever P(n) holds. Thus, by the principle of Mathematical Induction, $P(n)$ holds for all natural numbers n.

2 Euclid's Algorithm

(a) For any integer x, x divides 0, because $x \times 0 = 0$, therefore, a is a factor of 0 (written as $a|0$) and since, the greatest factor of a is a, therefore, $GCD(a, 0)$ is a.
(b) If, $a \leq b \Rightarrow GCD(b, a\%b) = GCD(b, a) = GCD(a, b)$,
Else if $a > b$, then let $a\%b = a - kb$, where k is the quotient when a is divided by b.
Let c be a common divisor of a and b.$\Rightarrow c|a, c|b \Rightarrow (a\%b)/c = a/c - k \times (b/c)$ is an integer because each term is an integer. $\Rightarrow c|(a\%b)$
Also, if c is a common divisor for b and $a\%b \Rightarrow c|b, c|(a\%b) \Rightarrow \frac{a}{c} = k \times \frac{b}{c} + \frac{(a\%b)}{c}$ is an integer, because all terms in the expansion are integers
$\Rightarrow c|a$
Thus, all common divisors of (a, b) and $(b, a\%b)$ are identical $\Rightarrow GCD(a, b) = GCD(a\%b, b)$
(c)**Case I**: If a $<$ 2b, then $(b + a\%b) \leq (a + b) - b \leq \frac{2}{3}(a + b)$
Case II: If $a \geq 2b$, then $(b + a\%b) \leq 2b \leq \frac{2}{3}(a + b)$
Thus, the value of $(a + b)$ reduces by a factor of at least $\frac{2}{3}$ in each step. T(a+b): Running time of the algorithm when the input is (a, b)
Induction Hypothesis P(N): $T(a + b) \leq \log_{\frac{3}{2}}(a + b) + k$, for some large k, to satisfy the base case.
is true for all $a + b \leq N$

Induction Step

$$
\begin{aligned}
T(a+b+1) &= 1 + T(\frac{2}{3}(a+b+1)) \\
&\leq 1 + \log_{\frac{3}{2}}(\frac{2}{3}(a+b+1)) + k && \text{(From the Induction Hypothesis)} \\
&= 1 + \log_{\frac{3}{2}}(a+b+1) + \log_{\frac{3}{2}}(\frac{2}{3}) + k && (log(ab) = log(a) + log(b)) \\
&= 1 + \log_{\frac{3}{2}}(a+b+1) - 1 + k && (\text{since } \log_{\frac{3}{2}}(\tfrac{2}{3}) = -1) \\
&= \log_{\frac{3}{2}}(a+b+1) + k
\end{aligned}
$$

Hence, by the principle of Mathematical Induction, P(a+b) holds true for all naturals a, b. Thus, T(a+b) = $\theta(log(a+b))$