Lecture 18 Union Find

Tuesday, November 7, 2017 2:27 PM

- (2^k+1)-th add operation
  running time 2^{k+1} "heavy" / "expensive"
  (allocating an array of size 2^{k+1}
  copy first 2^k elements
  add the (2^k+1)th element)
- all other add operation "light"
  running time 1 (change length, put the element into an empty slot)

$$- \begin{cases} T_{2^k+1} = 2^{k+1} & \text{for all } k \geq 0 \\ T_i = 1 & \text{for all } i \neq 2^k+1 \end{cases}$$

- Potential argument

$$A_i = T_i - \Phi(x_i) + \Phi(x_{i+1})$$
$$= T_i - \underbrace{(\Phi(x_i) - \Phi(x_{i+1}))}_{\text{potential difference}}$$

$$\begin{cases} A_1 = T_1 - \Phi(x_1) + \Phi(x_2) \\ A_2 = T_2 - \Phi(x_2) + \Phi(x_3) \\ \quad \cdots \\ A_n = T_n - \Phi(x_n) + \Phi(x_{n+1}) \end{cases}$$

$$\underbrace{\sum_{i=1}^{n} A_i}_{\text{total amortized}} = \underbrace{\sum_{i=1}^{n} T_i}_{\text{total runtime}} - \Phi(x_1) + \Phi(x_{n+1})$$

- design the potential function
  - idea: first make sure $\Phi(x_i) - \Phi(x_{i+1})$ is large
    if $i = 2^k+1$ is a heavy operation. $(T_i = 2^{k+1})$

Φ is a function of <u>\# elements</u> and <u>capacity</u>

Φ is a function of # elements and capacity

$$\underline{\#\text{ elements}} \qquad \underline{\text{capacity}}$$
$$\quad l \qquad\qquad\qquad c$$

	l	c
X_i	2^k	2^k

↑ before 2^k+1 add operation, data structure has 2^k elements.

X_{i+1}	2^k+1	2^{k+1}
difference	1	2^k

$\Rightarrow$ if capacity ↑ by 2^k, potential Φ ↓ by 2^{k+1}

$$\Phi = ? - 2\cdot c$$

- step2: $\Phi \geq 0$

observation: $l\cdot 2 \geq c$ (except for the initial state)

$1+2l \geq c$ (always true)

can use $\Phi = 2+4l-2c$

and we know $\Phi \geq 0$

- compute amortized running time

① heavy operation $i = 2^k+1$ $T_i = 2^{k+1}$

$$\Phi(X_i) = 2+4(2^k)-2(2^k)$$
$$\Phi(X_{i+1}) = 2+4(2^k+1)-2(2^{k+1})$$
$$\Phi(X_i)-\Phi(X_{i+1}) = 2^{k+1}-4$$
$$A_i = T_i - (\Phi(X_i)-\Phi(X_{i+1})) = 2^{k+1}-(2^{k+1}-4)$$
$$= 4$$

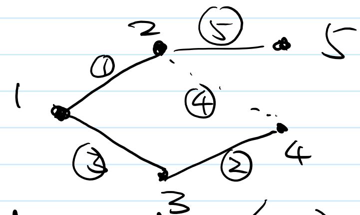
② light operation $i \neq 2^{k+1}$ $T_i = 1$

	l	c	Φ
X_i	$i-1$	c	$2+4(i-1)-2c$
X_{i+1}	i	c	$2+4i-2c$

$$\Phi(X_i)-\Phi(X_{i+1}) = -4$$
$$A_i = T_i - (\Phi(X_i)-\Phi(X_{i+1})) = 1-(-4) = 5$$

amortized cost = 5 = $O(1)$

- Union-Find
  - recall : Kruskal
    - for each edge : want to know whether adding the edge creates a cycle.
    - Sets $\Longleftrightarrow$ connected components



add an edge $\Longleftrightarrow$ union on the two connected components

edge creates a cycle (u,v) $\Longleftrightarrow$ Find(u) = Find(v)
u, v in the same set.

- edge ①: Find(1) = 1 Find(2) = 2 {1} {2} {3} {4} {5}
  union(1,2) {1, 2} {3} {4} {5}
- edge ② Find(3) = 3 Find(4) = 4
  union(3,4) {1, 2} {3, 4} {5}
- edge ③ (1,3) Find(1) = 1 Find(3) = 3
  union(1, 3) {1, 2, 3, 4} {5}
- edge ④ (2,4) Find(2) = 1 = Find(4) = 1
  do not add the edge