In this lecture we discuss *linear programming*, which is a very powerful tool that solve many problems.

1   Basic Definitions

A linear program tries to minimize a linear function under a set of linear constraints. It looks like

$$
\begin{aligned}
\max \quad & 2x + 3y \\
s.t. \quad & x + y && \leq 2 \\
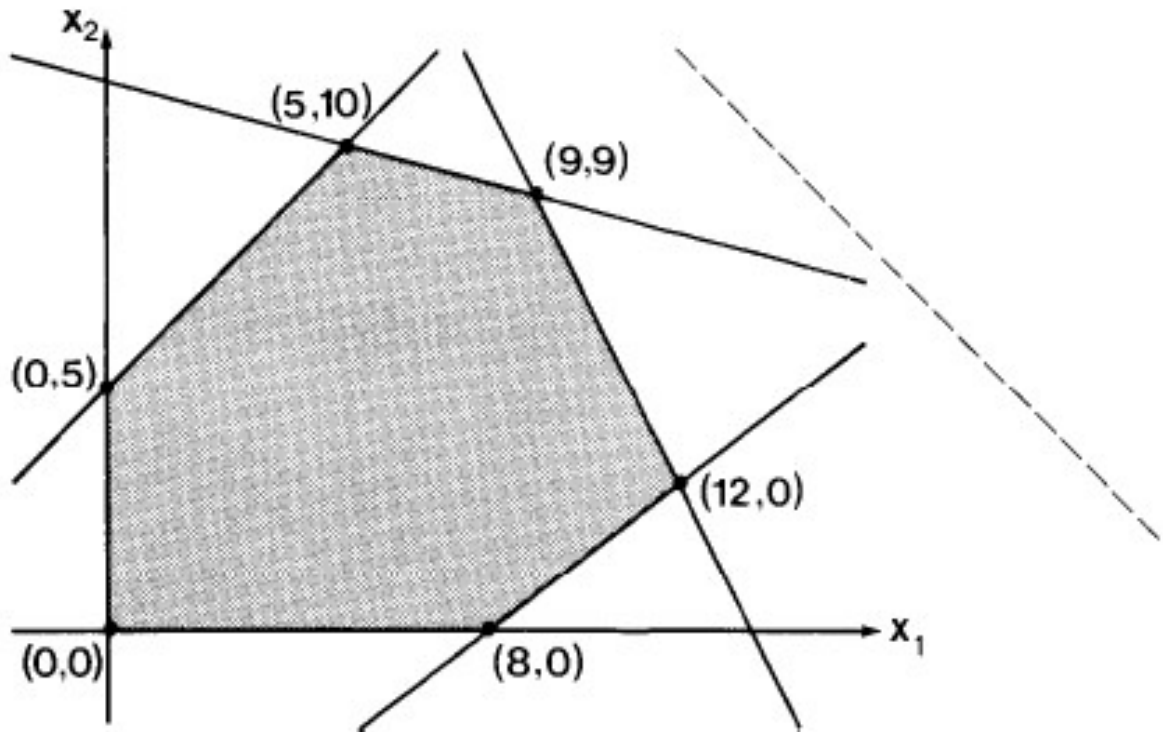& x - y && \geq 0 \\
& x && \geq 0
\end{aligned}
$$

Here $f(x, y) = 2x + 3y$ is the objective function, and the inequalities $x + y \leq 2, x - y \geq 0, x \geq 0$ are called constraints.

A feasible solution is a value for the variables (e.g. $(x, y) = (2, 0)$) that satisfies all the constraints. The value of a solution is the value of the objective function, so the value of $(2, 0)$ is equal to 4. We call the set of feasible solutions the feasible set.

The optimal solution is a solution that has the optimal value. In the above linear program, the optimal solution is $(x, y) = (1, 1)$, and the optimal value is 5.

2   Geometric Interpretation

We can interpret the constraints and objective of linear programs geometrically, see the figure below.

In the figure, the shaded area is the feasible set. Each constraint can be represented as a half-plane (for example, $x_1 \geq 0$ will correspond to the right half of the plane). The feasible set is just the intersection of these halfplanes.

In high dimensions, halfplanes become half-spaces and the feasible set is still the intersection of the half-spaces.

For the objective function, we can view it as a direction of gravity. The dashed line in the figure is orthogonal to this direction and should be considered as the "ground". If we rotate the figure so that the ground is at the bottom, then the optimal solution will correspond to the lowest point.

3   Canonical Form

In general, a linear program can have constraints of arbitrary forms as long as they are linear. However, to make it easier to implement/proof, we often want to convert linear programs to a *canonical form*:

The Standard Minimum Problem: Find an m-vector, $\mathbf{y} = (y_1, \ldots, y_m)$, to minimize

$$\mathbf{y}^{\mathsf{T}}\mathbf{b} = y_1 b_1 + \cdots + y_m b_m$$

subject to the constraints

$$y_1 a_{11} + y_2 a_{21} + \cdots + y_m a_{m1} \geq c_1$$
$$y_1 a_{12} + y_2 a_{22} + \cdots + y_m a_{m2} \geq c_2$$
$$\vdots$$
$$y_1 a_{1n} + y_2 a_{2n} + \cdots + y_m a_{mn} \geq c_n$$

(or $\mathbf{y}^T A \geq \mathbf{c}^T$)

and

$$y_1 \geq 0, y_2 \geq 0, \ldots, y_m \geq 0 \qquad (\text{or } \mathbf{y} \geq \mathbf{0}).$$

This can be represented the matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

Where n represents number of variables and m the number of constraints.

$$(Ay)_i = \sum_{j=1}^{n} a_{ij} y_j$$
$$b = (b_0, b_1 \ldots b_m)$$

4   convert LP to canonical form

There are several things we can do to convert a linear program into its canonical form.

The first idea is if there is an inequality/objective of the incorrect direction, then we can multiply it by -1 and change its direction

$$\max 2x + y$$
$$x \geq 0$$
$$y \geq 0$$
$$x + y \leq 1$$

Say $x_1 = x, x_2 = y$. We get:

$$\min -2x_1 - x_2$$
$$x + y \leq 1 \Rightarrow -x_1 - x_2 \geq 1$$
$$x_1 \geq 0$$
$$x_2 \geq 0$$

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In this case, $A = (-1, -1), b = (-1)$

Another case is if we have equality constraints. We can simply split the equality constraint into two inequality constraints ($A = B$ if and only if $A \geq B$ and $A \leq B$). We may also have a variable x that is not necessarily non-negative, in this case we split x into two variables $x_1 - x_2$, and require $x_1, x_2 \geq 0$.

The following example illustrates these two cases:

$$\min y + 2z$$
$$z \geq 0 \ \ y + z = 5$$

We split y into two variables: $y = x_1 - x_2$, $x_1 \geq 0$, $x_2 \geq 0$. Let $z = x_3$, we get

$$\min x_1 - x_2 + 2x_3$$
$$x_1, x_2, x_3 \geq 0$$
$$x_1 - x_2 + x_3 = 5$$

We can then rewrite the equality constraint into two inequality constraints:

$$x_1 - x_2 + x_3 = 5$$
$$x_1 - x_2 + x_3 \geq 5$$
$$x_1 - x_2 + x_3 \leq 5$$
$$\rightarrow -x_1 + x_2 - x_3 \geq -5$$

The final linear program we get is

$$\min x_1 - x_2 + 2x_3$$
$$x_1, x_2, x_3 \geq 0$$
$$x_1 - x_2 + x_3 \geq 5$$
$$-x_1 + x_2 - x_3 \geq -5$$

$$A = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \end{pmatrix} b = \begin{pmatrix} 5 \\ -5 \end{pmatrix} c = \begin{pmatrix} 1 & -1 & 2 \end{pmatrix}$$

5   applying linear program

Matching Problem: For each edge have an indicator variable x_{ij} which is 1 if (i, j) is matched, 0 if not.

Constraints:

- $0 \leq x_{ij} \leq 1$

- Every class has one course: $\forall j \sum_{i:(i,j) \in E} x_{ij} \leq 1$

- Every corse has ≤ 1 classroom$\forall i \sum_{j:(i,j) \in E} x_{ij} \leq 1$

Objective: Match as many pairs as possible

$$\max \sum_{(i,j) \in E} x_{ij}$$

This can be solved with an LP. If you assume x_{ij} are 0 or 1, then it is matching problem, if not (can be fractional), then the solution does not correspond to a matching.

For matching problem, we can prove that the LP will always give an integral solution (all x_{ij}'s are either 0 or 1).