Lecture 21 Linear Program (cont'd)

Thursday, November 16, 2017 2:35 PM

$$\min\ 2x_1 - 3x_2 + x_3$$

$x_1 - x_2 \geq 1$ (1) $\times$ y_1

$x_2 - 2x_3 \geq 2$ (2) $\times$ y_2

$-x_1 - x_2 - x_3 \geq -7$ (3) $\times$ y_3

$x_1, x_2, x_3 \geq 0$

Q: how to prove optimal value ≥ -1

$$\underline{2x_1 - 3x_2 + x_3 \geq -1}$$

strategy: use a linear combination of the constraints

- How to prove new inequalities?

(1) $2x_1 - x_2 + x_3 \geq 1$

$$2x_1 - x_2 + x_3 \underset{\substack{\uparrow \\ x_1, x_2, x_3 \geq 0}}{\geq} x_1 - x_2 \underset{\substack{\uparrow \\ \text{constraint (1)}}}{\geq} 1$$

general form: if $a_1 \geq b_1$, $a_2 \geq b_2$, $a_3 \geq b_3$, $x_1, x_2, x_3 \geq 0$

$$a_1 x_1 + a_2 x_2 + a_3 x_3 \geq b_1 x_1 + b_2 x_2 + b_3 x_3$$

(2) $x_1 - 2x_3 \geq 3$

$x_1 - x_2 \geq 1$ (1), $x_2 - 2x_3 \geq 2$ (2)

(1) + (2)

$$x_1 - 2x_3 = x_1 - x_2 + x_2 - 2x_3 \geq 1 + 2 = 3$$

Proof: $x_1 - x_2 \geq 1$ (1)

$-x_1 - x_2 - x_3 \geq -7$ (3)

$2.5 \times (1) + 0.5 \times (3)$ (2)

$$2.5x_1 - 2.5x_2 + 0.5(-x_1 - x_2 - x_3) \geq 2.5 + 0.5(-7)$$

$$2x_1 - 3x_2 - 0.5x_3 \geq -1$$

(recall: want $2x_1 - 3x_2 + x_3 \geq -1$)

$$2x_1 - 3x_2 + x_3 \geq 2x_1 - 3x_2 - 0.5x_3 \geq -1$$ (1)

- Question: How to choose the coefficients to multiply the constraints with?

$\min\ 2x_1 - 3x_2 + x_3$

$$x_1 - x_2 \geq 1 \quad (1) \times \quad y_1$$
$$x_2 - 2x_3 \geq 2 \quad (2) \times \quad y_2$$
$$-x_1 - x_2 - x_3 \geq -7 \quad (3) \times \quad y_3$$
$$x_1, x_2, x_3 \geq 0$$

$- \ y_1, y_2, y_3 \geq 0$

$$y_1 \times (1) + y_2 \times (2) + y_3 \times (3)$$

$$\underbrace{(y_1 - y_3)x_1 + (-y_1 + y_2 - y_3)x_2 + (-2y_2 - y_3)x_3}_{LHS} \geq y_1 + 2y_2 - 7y_3$$

want: $2x_1 - 3x_2 + x_3 \underset{?}{\geq} LHS \geq y_1 + 2y_2 - 7y_3$

$$y_1 - y_3 \leq 2$$
$$-y_1 + y_2 - y_3 \leq -3$$
$$-2y_2 - y_3 \leq 1$$

want the strongest inequality $\Longrightarrow$ $\max\ y_1 + 2y_2 - 7y_3$

- Theorem: Strong duality optimal value of primal and dual are equal.
- Complimentary slackness.

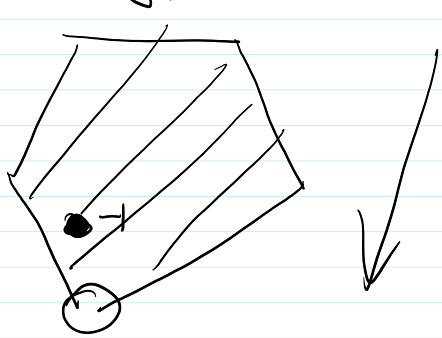
  if $x = (x_1, \cdots, x_n)$ is a primal optimal solution

  $y = (y_1, \cdots, y_m)$ is a dual optimal solution

  if $a_1x_1 + a_2x_2 + \cdots + a_nx_n \geq b$ is the i-th constraint

  if $a_1x_1 + a_2x_2 + \cdots + a_nx_n > b$ (constraint not tight)

  then $y_i = 0$

- 

- shortest path as a linear program

  0 edge not

- shortest path as a linear program

variables: for each edge (u,v) $X_{u,v} = \begin{cases} 0 & \text{edge not in shortest path} \\ 1 & \text{edge in shortest path.} \end{cases}$

$$0 \leq X_{u,v} \leq 1$$

- constraints: $X_{u,v}$ form a path from s to t.

for intermediate vertices: # incoming edges = # outgoing edges

for any v, $v \neq s,t$: $\underbrace{\sum_u X_{u,v}}_{\#\text{ incoming}} = \underbrace{\sum_u X_{v,u}}_{\#\text{ outgoing}}$

for s: $\sum_u X_{s,u} = 1$ $\quad \sum_u X_{u,s} = 0$

t: $\sum_u X_{t,u} = 0$ $\quad \sum_u X_{u,t} = 1$

- objective $\min \underbrace{\sum_{(u,v) \in E} X_{u,v} \cdot W_{u,v}}_{\text{total length}}$