Extensive-form games

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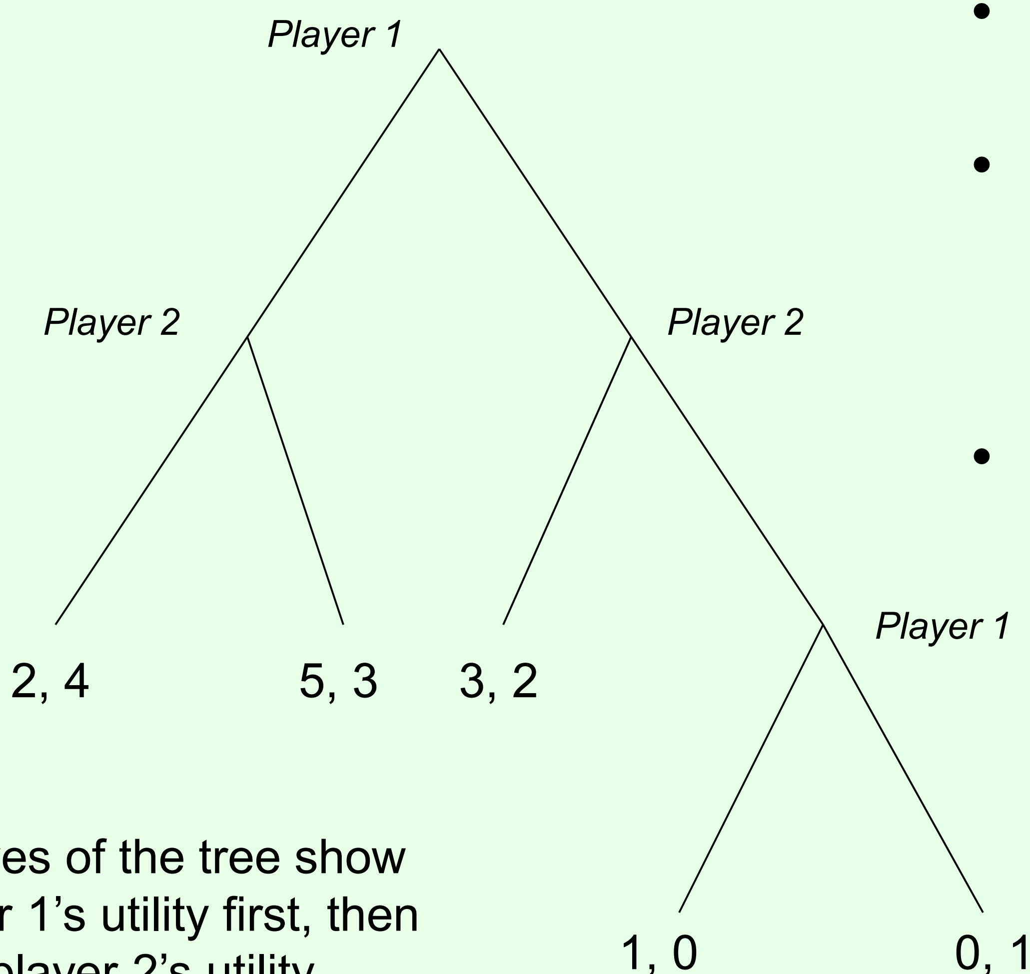
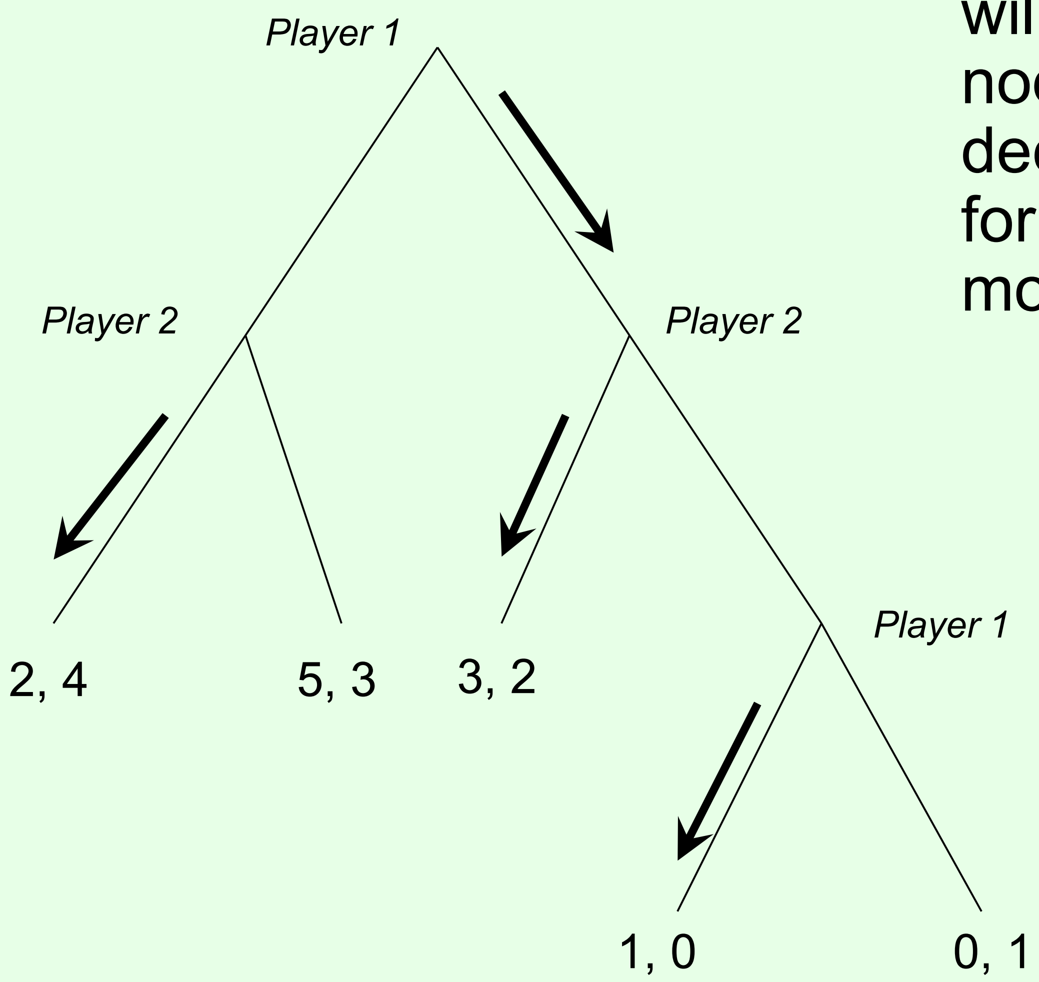
Extensive-form games with perfect information



Leaves of the tree show player 1's utility first, then player 2's utility

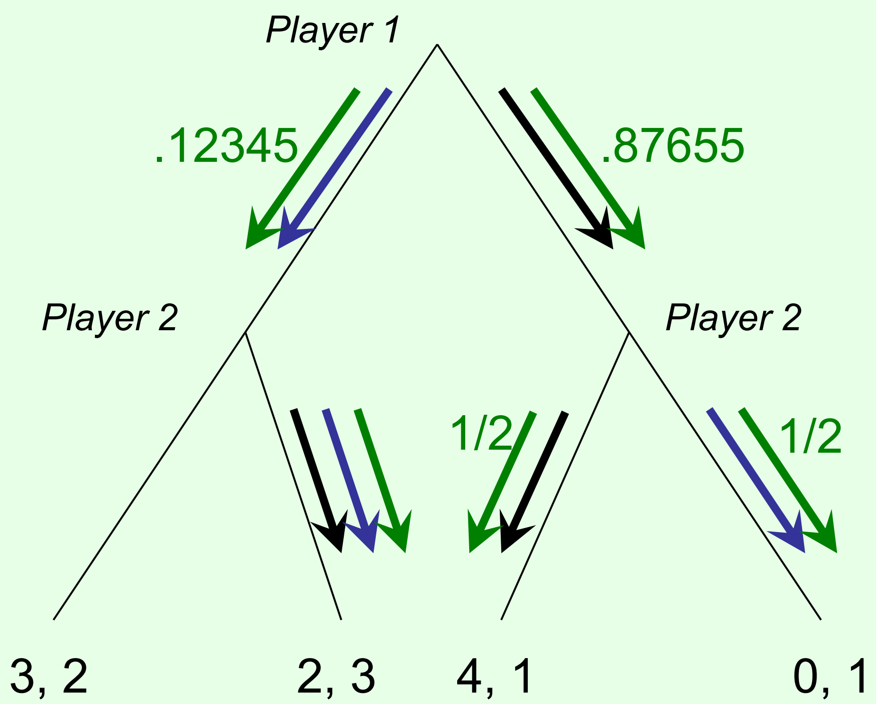
- Players do not move simultaneously
- When moving, each player is aware of all the previous moves (perfect information)
- A (pure) strategy for player i is a mapping from player i's nodes to actions

Backward induction



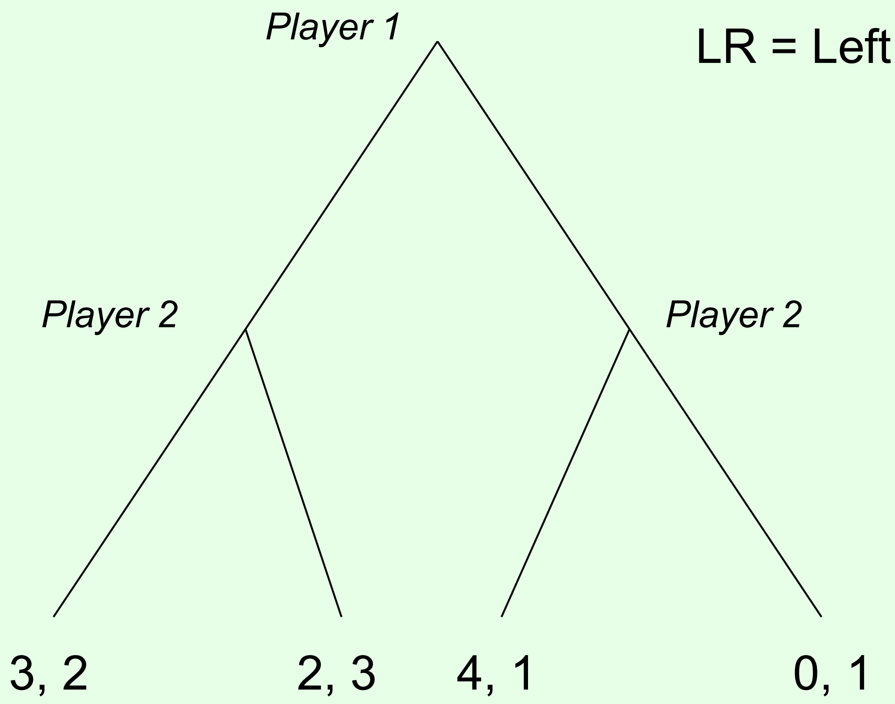
- When we know what will happen at each of a node's children, we can decide the best action for the player who is moving at that node

A limitation of backward induction



- If there are ties, then how they are broken affects what happens higher up in the tree
- Multiple equilibria…

Conversion from extensive to normal form

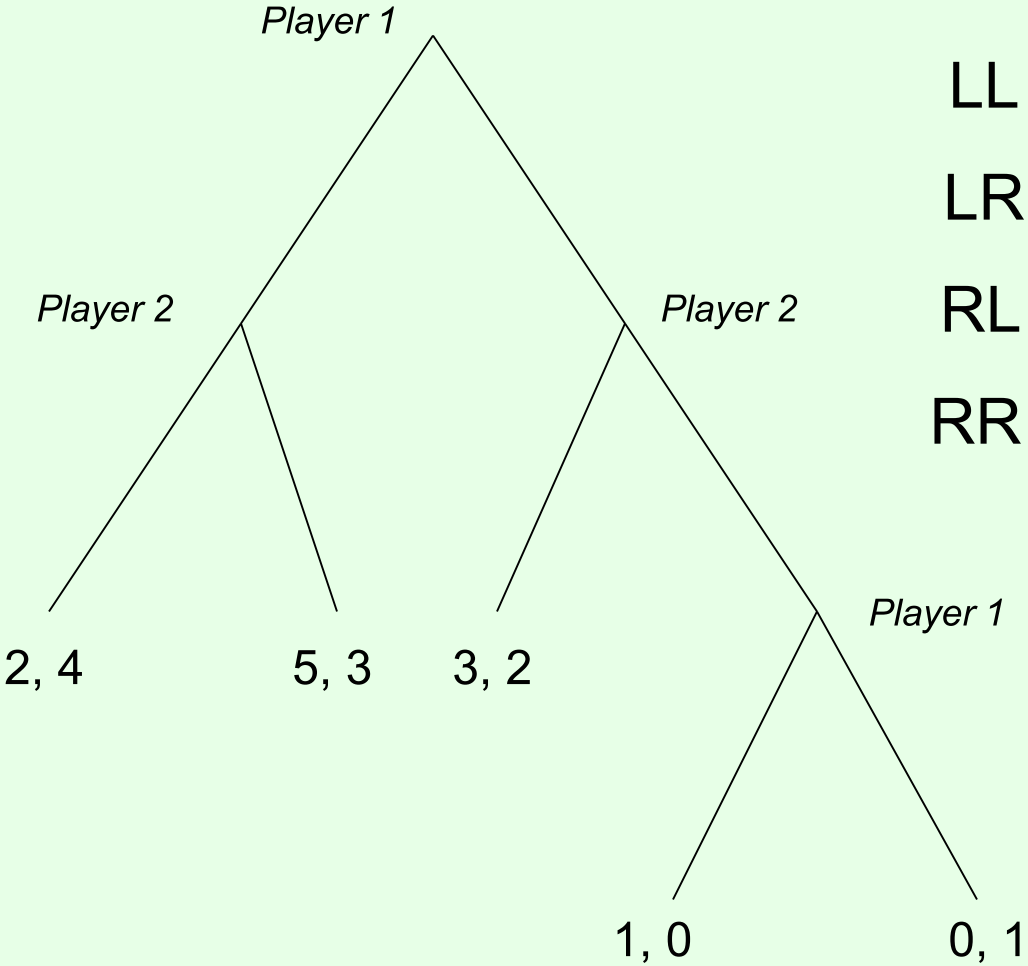


LR = Left if 1 moves Left, Right if 1 moves Right; etc.

	LL	LR	RL	RR
L	3, 2	3, 2	2, 3	2, 3
R	4, 1	0, 1	4, 1	0, 1

- Nash equilibria of this normal-form game include (R, LL), (R, RL), (L, RR) + infinitely many mixed-strategy equilibria
- In general, normal form can have exponentially many strategies

Converting the first game to normal form

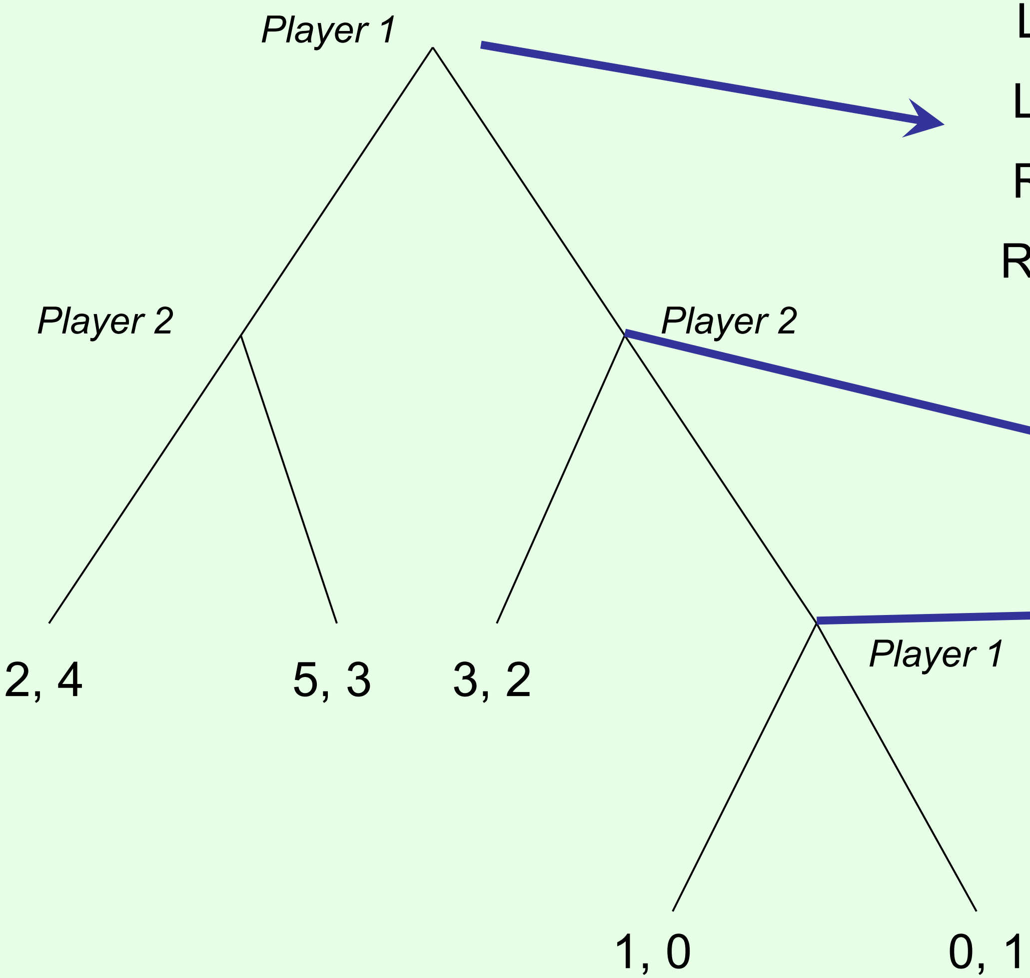


	LL	LR	RL	RR
LL	2, 4	2, 4	5, 3	5, 3
LR	2, 4	2, 4	5, 3	5, 3
RL	3, 2	1, 0	3, 2	1, 0
RR	3, 2	0, 1	3, 2	0, 1

- Pure-strategy Nash equilibria of this game are (LL, LR), (LR, LR), (RL, LL), (RR, LL)
- But the only backward induction solution is (RL, LL)

Subgame perfect equilibrium

- Each node in a (perfect-information) game tree, together with the remainder of the game after that node is reached, is called a subgame
- A strategy profile is a subgame perfect equilibrium if it is an equilibrium for every subgame



	LL	LR	RL	RR
LL	2, 4	2, 4	5, 3	5, 3
LR	2, 4	2, 4	5, 3	5, 3
RL	3, 2	1, 0	3, 2	1, 0
RR	3, 2	0, 1	3, 2	0, 1

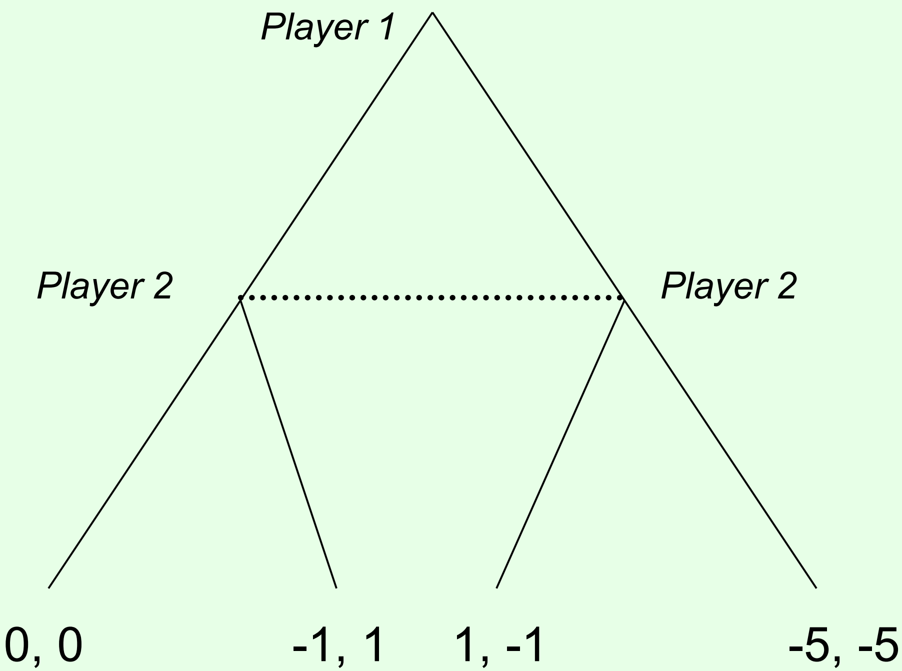
	*L	*R
*L	3, 2	1, 0
*R	3, 2	0, 1

	**
*L	1, 0
*R	0, 1

- (RR, LL) and (LR, LR) are not subgame perfect equilibria because (*R, **) is not an equilibrium
- (LL, LR) is not subgame perfect because (*L, *R) is not an equilibrium
  - *R is not a credible threat

Imperfect information

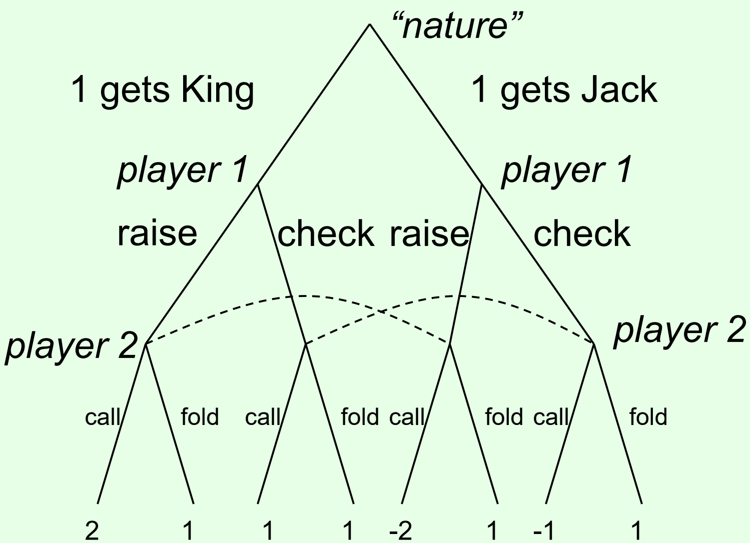
- Dotted lines indicate that a player cannot distinguish between two (or more) states
  - A set of states that are connected by dotted lines is called an information set
- Reflected in the normal-form representation



	L	R
L	0, 0	-1, 1
R	1, -1	-5, -5

- Any normal-form game can be transformed into an imperfect-information extensive-form game this way

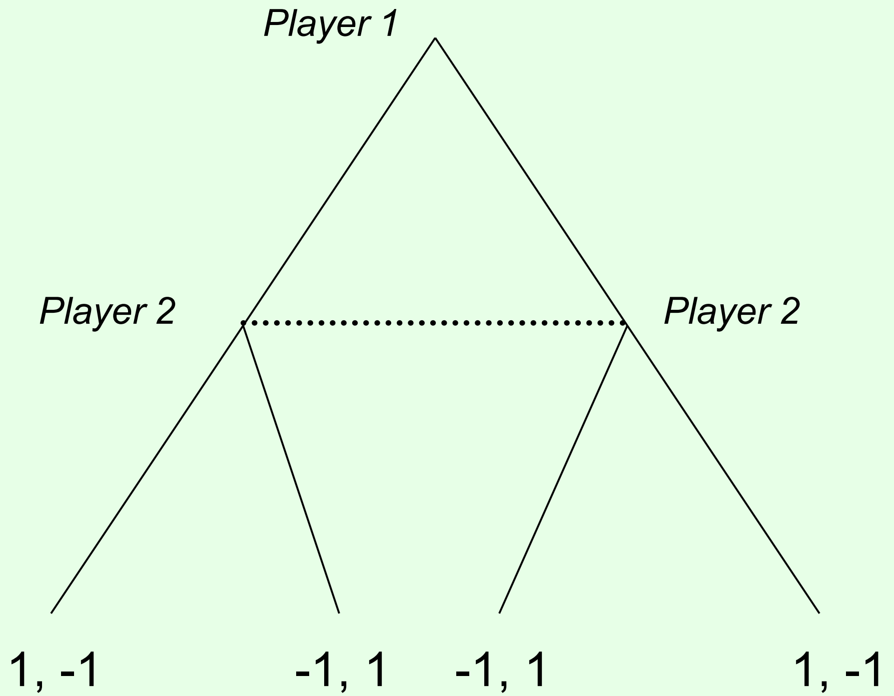
A poker-like game



		2/3 cc	cf	1/3 fc	ff
1/3	rr	0, 0	0, 0	1, -1	1, -1
2/3	rc	.5, -.5	1.5, -1.5	0, 0	1, -1
	cr	-.5, .5	-.5, .5	1, -1	1, -1
	cc	0, 0	1, -1	0, 0	1, -1

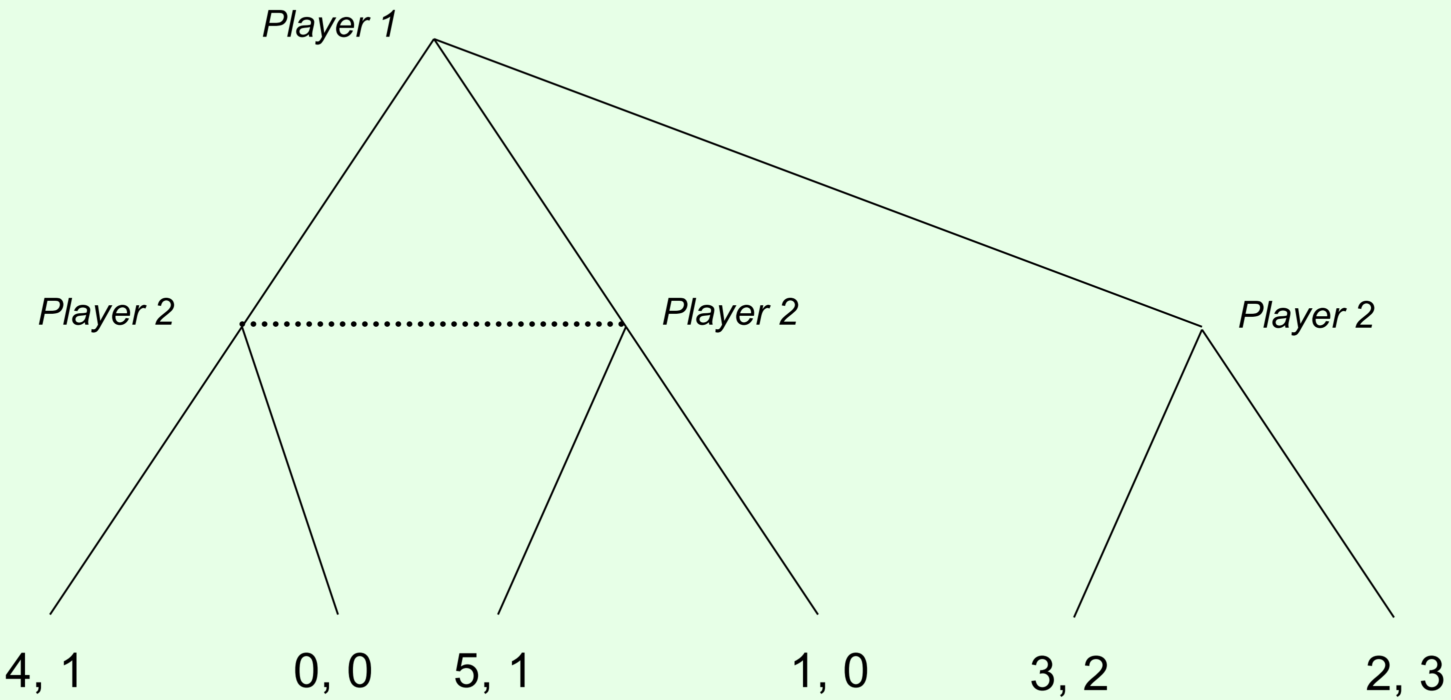
Subgame perfection and imperfect information

- How should we extend the notion of subgame perfection to games of imperfect information?



- We cannot expect Player 2 to play Right after Player 1 plays Left, and Left after Player 1 plays Right, because of the information set
- Let us say that a subtree is a subgame only if there are no information sets that connect the subtree to parts outside the subtree

Subgame perfection and imperfect information…



- One of the Nash equilibria is: (R, RR)
- Also subgame perfect (the only subgames are the whole game, and the subgame after Player 1 moves Right)
- But it is not reasonable to believe that Player 2 will move Right after Player 1 moves Left/Middle (not a credible threat)
- There exist more sophisticated refinements of Nash equilibrium that rule out such behavior

Computing equilibria in the extensive form

- Can just use normal-form representation
  - Misses issues of subgame perfection, etc.
- Another problem: there are exponentially many pure strategies, so normal form is exponentially larger
  - Even given polynomial-time algorithms for normal form, time would still be exponential in the size of the extensive form
- There are other techniques that reason directly over the extensive form and scale much better
  - E.g., using the sequence form of the game

Commitment

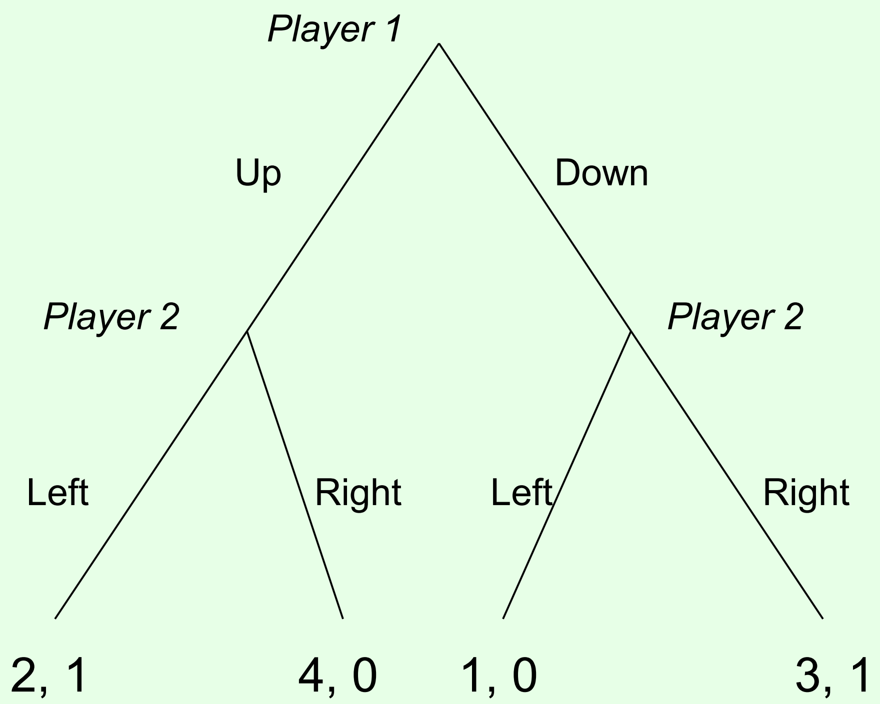
- Consider the following (normal-form) game:

2, 1	4, 0
1, 0	3, 1

- How should this game be played?
- Now suppose the game is played as follows:
  - Player 1 commits to playing one of the rows,
  - Player 2 observes the commitment and then chooses a column
- What is the optimal strategy for player 1?
- What if 1 can commit to a mixed strategy?

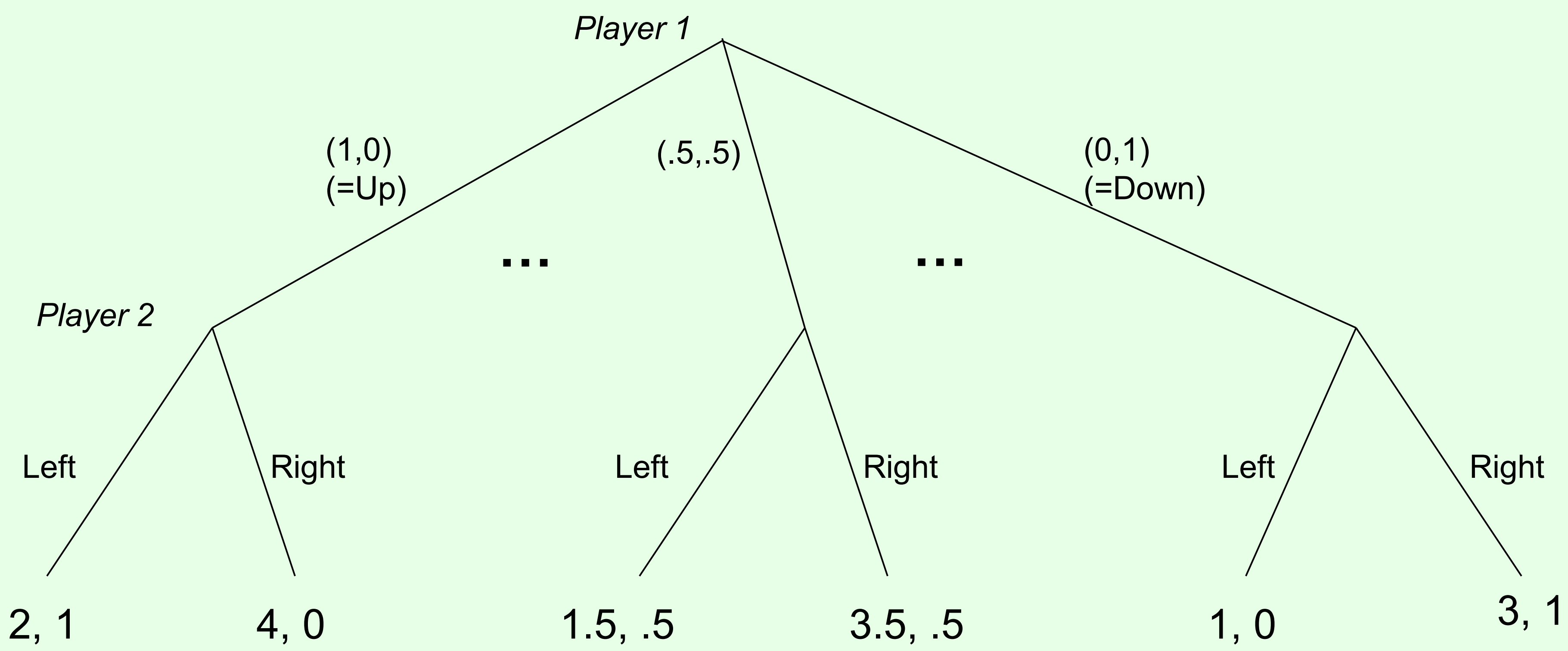
Commitment as an extensive-form game

- For the case of committing to a pure strategy:



Commitment as an extensive-form game

- For the case of committing to a mixed strategy:



- Infinite-size game; computationally impractical to reason with the extensive form here

Solving for the optimal mixed strategy to commit to

[Conitzer & Sandholm 2006, von Stengel & Zamir 2010]

- For every column t separately, we will solve separately for the best mixed row strategy (defined by $\mathbf{p_s}$) that induces player 2 to play t
- maximize $\Sigma_s \mathbf{p_s} u_1(s, t)$
- subject to

  for any t', $\Sigma_s \mathbf{p_s} u_2(s, t) \geq \Sigma_s \mathbf{p_s} u_2(s, t')$

  $\Sigma_s \mathbf{p_s} = 1$
- (May be infeasible, e.g., if t is strictly dominated)
- Pick the t that is best for player 1

Visualization

	L	C	R
U	0,1	1,0	0,0
M	4,0	0,1	0,0
D	0,0	1,0	1,1